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A Novel Similarity Measure for Generalized Nonlinear Trapezoidal Fuzzy Numbers with Applications to Multi-Attribute Decision Making

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ABSTRACT

There are several similarity measure techniques for generalized trapezoidal fuzzy numbers. However, no similarity measure technique has yet been proposed for generalized non-linear trapezoidal fuzzy numbers. In the present work, a new similarity measure approach for two generalized non-linear trapezoidal fuzzy numbers is proposed based on distance and center of gravity (COG) measures. Several propositions related to the proposed method are also discussed. Furthermore, a comparative analysis with existing methods is carried out through illustrative examples, demonstrating that the proposed method is applicable to both generalized non-linear and generalized linear trapezoidal fuzzy numbers. Finally, a multi-attribute decision-making problem is considered, and a new algorithm is introduced to identify the best alternative for a decision maker using the proposed similarity measure method.

1. Introduction

To model uncertainty in real-life applications, fuzzy set theory has played a significant role since its introduction by Zadeh in 1965 [1]. Since then, numerous properties and applications of fuzzy sets have been developed by researchers in various fields. In a crisp set, an element either belongs to the set or does not belong to it. In contrast, in a fuzzy set, each element is associated with a membership value. Therefore, a fuzzy set is represented by ordered pairs consisting of elements and their corresponding membership values.

Fuzzy numbers represent membership values corresponding to uncertain quantities. Various types of fuzzy numbers have been introduced in the literature, such as triangular fuzzy numbers, trapezoidal fuzzy numbers, Gaussian fuzzy numbers, and bell-shaped fuzzy numbers. Among these, trapezoidal

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fuzzy numbers are one of the most widely used due to their simplicity and computational efficiency. A generalized trapezoidal fuzzy number is an extension of the classical trapezoidal fuzzy number.

Unlike real numbers, fuzzy numbers cannot be ordered directly. Therefore, it is necessary to compare fuzzy numbers to determine how close they are to each other. Similarity measures provide an effective tool for this purpose. Various similarity measure methods have been proposed for comparing trapezoidal fuzzy numbers. Chen and Chen [2] introduced a similarity measure based on the concept of center of gravity (COG) distance. Chen and Chen [2] further studied fuzzy risk analysis using similarity measures of generalized fuzzy numbers. Wei and Chen [3] proposed a similarity measure for generalized trapezoidal fuzzy numbers and applied it to fuzzy risk analysis using linguistic term values. Xu et al. [4] introduced a similarity measure based on the COG points of linguistic-valued trapezoidal fuzzy numbers and proposed a new arithmetic operator for linguistic-valued trapezoidal fuzzy numbers. Hejazi et al. [5] developed a similarity measure between generalized trapezoidal fuzzy numbers based on area and perimeter. Chen and Sanguansat [6] presented a method for measuring the similarity between interval-valued trapezoidal fuzzy numbers. Patra and Mondal [7] proposed a similarity measure based on area and height and applied it to a risk analysis problem. Sen et al. [8] introduced a similarity measure for interval-valued fuzzy numbers and applied it to the fuzzy risk analysis of familial breast cancer. Similarity-based risk analysis has also been studied in production inventory models and supplier selection problems. Chen et al. [9] presented a similarity measure between IFSs based on the centroid points of transformed fuzzy numbers for pattern recognition problems. Sen et al. [10] proposed a similarity measure for generalized trapezoidal fuzzy numbers and applied it to a risk analysis problem. Sen et al. [11] developed a similarity measure for Gaussian fuzzy numbers and applied it to a candidate selection problem. Dutta and Borah [12] proposed a similarity measure for generalized trapezoidal fuzzy numbers and applied it to a multi-criteria decision-making problem.

Several ranking methods have also been developed for comparing fuzzy numbers. Abbasbandy and Hajjari [13] proposed a ranking technique in 2009. Chen and Sanguansat [6] introduced a ranking method and applied it to a risk analysis problem. Patra and Mondal [14] proposed a ranking method based on the area, mean position, and range of fuzzy numbers and applied it to diabetes risk analysis. Chutia and Chutia [15] developed a ranking method based on value and ambiguity. Chutia [16] further proposed a ranking method using value and angle in the epsilon-deviation degree framework. Patra [17] introduced a ranking method for interval-valued intuitionistic fuzzy values and applied it to a multi-attribute decision-making problem.

Multi-attribute decision-making (MADM) and multi-criteria decision-making (MCDM) problems are inherently vague and uncertain. Therefore, crisp data are often unavailable in practical applications. As a result, various MADM/MCDM methods have been developed using different types of fuzzy numbers. Li [18] applied a TOPSIS-based nonlinear programming methodology to MADM problems under IVIFSs. Chen et al. [19] proposed a MADM method based on IVIFSs. Mishra et al. [20] established an extended MABAC method based on divergence measures for multi-criteria assessment of programming languages under IVIFSs. Chen and Han [21] presented a MADM method integrating nonlinear programming (NLP), particle swarm optimization (PSO), and IVIFVs.

All existing similarity measure methods have been developed for linear fuzzy numbers. To the best of our knowledge, no similarity measure method has been proposed for generalized non-linear trapezoidal fuzzy numbers. Therefore, in the present work, a new similarity measure approach is developed for generalized non-linear trapezoidal fuzzy numbers. To the best of our knowledge, this is the first study introducing such a similarity measure technique. Several propositions related to the proposed similarity measure are also established. Although no similarity measure currently exists for generalized non-linear trapezoidal fuzzy numbers, several examples are considered to compare the proposed method with existing similarity measures for generalized linear trapezoidal fuzzy numbers. The results demonstrate that existing methods fail to accurately evaluate similarity in the case of generalized non-linear trapezoidal fuzzy numbers. Consequently, the proposed method provides an effective tool for evaluating similarity between both generalized linear and generalized non-linear trapezoidal fuzzy numbers.

The remainder of the paper is organized as follows. Section 2 presents the necessary preliminaries. Section 3 introduces the proposed similarity measure for generalized non-linear trapezoidal fuzzy numbers. Section 4 presents several propositions related to the proposed method. Section 5 provides illustrative examples to demonstrate the effectiveness of the proposed approach. Section 6 presents a comparative analysis with existing methods. Finally, Section 7 concludes the paper.

2. Preliminaries

Definition 1. ***Fuzzy Set:*** *[1]*

A fuzzy set $\tilde{A}$ in the universe of discourse X is defined by

$$\tilde{A} = \{(x, \mu_A(x)) | x \in X\}$$

where $\mu_A(x)$ denote the membership grade of x to $\tilde{A}$, $x \in X$, $0 \leq \mu_A(x) \leq 1$.

Definition 2. ***Generalized Trapezoidal Fuzzy Number:***

A trapezoidal fuzzy number $\tilde{A} = (a, b, c, d; w)$ is said to be generalized trapezoidal fuzzy number if $0 \leq a \leq b \leq c \leq d \leq 1$ and w is lying between 0 and 1.
The membership function of a generalized trapezoidal fuzzy number is denoted by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & if -\propto < x \leq a \\ f_A^L(x), & if\ a < x \leq b \\ w, & if\ b < x \leq c \\ f_A^R(x), & if\ c < x \leq d \\ 0, & if\ d < x < \propto \end{cases}$$

where $f_A^L(x) = \frac{w(x-a)}{b-a}$ and $f_A^R(x) = \frac{w(d-x)}{d-c}$.

Definition 3. ***Generalized Non Linear Trapezoidal Fuzzy Number:***

A trapezoidal fuzzy number $\tilde{A} = (a, b, c, d; w, p)$ is said to be a generalized non-linear trapezoidal fuzzy number if $0 \leq a \leq b \leq c \leq d \leq 1$, $0 \leq w \leq 1$, and p is any positive real number. The membership function of a generalized non-linear trapezoidal fuzzy number is defined as

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & if -\propto < x \leq a \\ \left(f_A^L(x)\right)^p, & if\ a < x \leq b \\ w, & if\ b < x \leq c \\ \left(f_A^R(x)\right)^p, & if\ c < x \leq d \\ 0, & if\ d < x < \propto \end{cases}$$

where $f_A^L(x) = \frac{w(x-a)}{b-a}$ and $f_A^R(x) = \frac{w(d-x)}{d-c}$. $g_{\tilde{A}}(x)$ denotes the inverse function of $\mu_{\tilde{A}}(x)$.

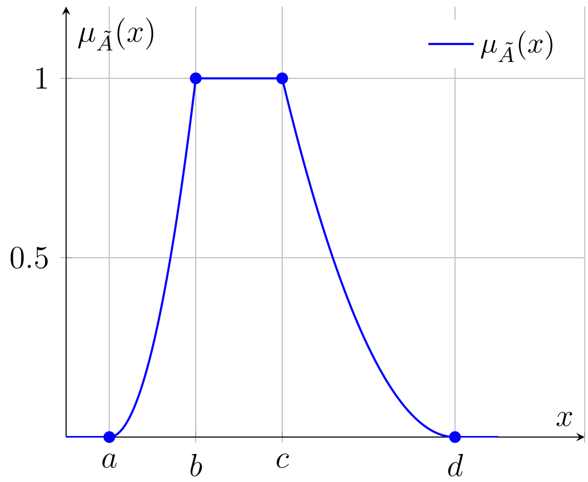


Fig. 1. Graphical representation of Non-Linear Generalized Fuzzy Numbers

3. New approach similarity measure:

Let $\tilde{X}_1 = (a_1, a_2, a_3, a_4; w_1, q_1)$ and $\tilde{X}_2 = (b_1, b_2, b_3, b_4; w_2, q_2)$ are two generalized non linear fuzzy numbers where $0 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq 1, 0 \leq b_1 \leq b_2 \leq b_3 \leq b_4 \leq 1, 0 \leq w_1, w_2 \leq 1$ and q_1, q_2 be any positive real numbers. Then the similarity measure between these fuzzy numbers have been calculated as follows:

Step 1: Calculate the distance measure of the fuzzy numbers $\tilde{X}_1$ and $\tilde{X}_2$ as

$$D_1(\tilde{X}_1, \tilde{X}_2) = \left(1 - \frac{1}{4}\sum_{i=1}^{4}|a_i - b_i|\right)$$

Step 2: Calculate the COG points (x_{i1}, y_{i1}) of the fuzzy numbers $\tilde{X}_1$ for each subinterval $[a_1, a_2]$, $[a_2, a_3]$ and $[a_3, a_4]$ as follows

$$x_{i1} = \frac{\displaystyle\int_{a_i}^{a_{i+1}} x\mu_{X_1}(x)}{\displaystyle\int_{a_i}^{a_{i+1}} \mu_{X_1}(x)} \text{ and } y_{i1} = \frac{\displaystyle\int_{0}^{w_1} y g_{X_1}(y)}{\displaystyle\int_{0}^{w_1} g_{X_1}(y)} \qquad i = 1, 2, 3.$$

Step 3: Calculate the COG points (x_{i2}, y_{i2}) of the fuzzy numbers $\tilde{X}_2$ for each subinterval $[b_1, b_2]$, $[b_2, b_3]$ and $[b_3, b_4]$ as follows

$$x_{i2} = \frac{\displaystyle\int_{b_i}^{b_{i+1}} x\mu_{X_2}(x)}{\displaystyle\int_{b_i}^{b_{i+1}} \mu_{X_2}(x)} \text{ and } y_{i2} = \frac{\displaystyle\int_{0}^{w_2} y g_{X_2}(y)}{\displaystyle\int_{0}^{w_2} g_{X_2}(y)} \qquad i = 1, 2, 3.$$

Step 4: Calculate the COG distance measure of the fuzzy numbers as

$$D_2(\tilde{X}_1, \tilde{X}_2) = \left(1 - \frac{1}{3}\sqrt{\sum_{i=1}^{3}\left\{(x_{i1} - x_{i2})^2 + (y_{i1} - y_{i2})^2\right\}}\right)$$

Step 5: The similarity measure between two fuzzy numbers is calculated as

$$S(\tilde{X}_1, \tilde{X}_2) = D_1(\tilde{X}_1, \tilde{X}_2) \times D_2(\tilde{X}_1, \tilde{X}_2) \tag{1}$$

4. Propositions

Proposition 1. $D_1(\tilde{X}_1, \tilde{X}_2) \in [0,1]$ *and* $D_2(\tilde{X}_1, \tilde{X}_2) \in [0,1]$.

Proof. Let $\tilde{X}_1 = (a_1, a_2, a_3, a_4; w_1, q_1)$ and $\tilde{X}_2 = (b_1, b_2, b_3, b_4; w_2, q_2)$ are two generalized non linear fuzzy numbers where $0 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq 1$, $0 \leq b_1 \leq b_2 \leq b_3 \leq b_4 \leq 1$, $0 \leq w_1, w_2 \leq 1$ and q_1, q_2 be any positive real numbers. Then $D_1(\tilde{X}_1, \tilde{X}_2) = \big(1 - \frac{1}{4}\sum_{i=1}^{4} |a_i - b_i|\big)$.
Now, $0 \leq |a_i - b_i| \leq 1$, so, $D_1(\tilde{X}_1, \tilde{X}_2) \in [0,1]$.
Since, $0 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq 1$, $0 \leq b_1 \leq b_2 \leq b_3 \leq b_4 \leq 1$, $0 \leq w_1, w_2 \leq 1$, so, $x_{i1}, y_{i1}, x_{i2}, y_{i2} \in [0,1]$.
Therefore, $\boxed{D_2(\tilde{X}_1, \tilde{X}_2) \in [0,1]}$. □

Proposition 2. $D_1(\tilde{X}_1, \tilde{X}_2) = D_1(\tilde{X}_2, \tilde{X}_1)$ *and* $D_2(\tilde{X}_1, \tilde{X}_2) = D_2(\tilde{X}_2, \tilde{X}_1)$.

Proof. Let $\tilde{X}_1 = (a_1, a_2, a_3, a_4; w_1, q_1)$ and $\tilde{X}_2 = (b_1, b_2, b_3, b_4; w_2, q_2)$ are two generalized non linear fuzzy numbers where $0 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq 1$, $0 \leq b_1 \leq b_2 \leq b_3 \leq b_4 \leq 1$, $0 \leq w_1, w_2 \leq 1$ and q_1, q_2 be any positive real number. Then

$$D_1(\tilde{X}_2, \tilde{X}_1) = \left(1 - \frac{1}{4}\sum_{i=1}^{4} |b_i - a_i|\right) = \left(1 - \frac{1}{4}\sum_{i=1}^{4} |a_i - b_i|\right) = D_1(\tilde{X}_1, \tilde{X}_2)$$

i.e., $\boxed{D_1(\tilde{X}_1, \tilde{X}_2) = D_1(\tilde{X}_2, \tilde{X}_1)}$.

$$\begin{aligned} D_2(\tilde{X}_1, \tilde{X}_2) &= \left(1 - \frac{1}{3}\sqrt{\sum_{i=1}^{3}\left\{(x_{i1} - x_{i2})^2 + (y_{i1} - y_{i2})^2\right\}}\right) \\ &= \left(1 - \frac{1}{3}\sqrt{\sum_{i=1}^{3}\left\{(x_{i2} - x_{i1})^2 + (y_{i2} - y_{i1})^2\right\}}\right) \\ &= D_2(\tilde{X}_2, \tilde{X}_1) \end{aligned}$$

i.e., $\boxed{D_2(\tilde{X}_1, \tilde{X}_2) = D_2(\tilde{X}_2, \tilde{X}_1)}$. □

Proposition 3. $D_1(\tilde{X}_1, \tilde{X}_2) = 1$ *and* $D_2(\tilde{X}_1, \tilde{X}_2) = 1$ *iff* $\tilde{X}_1 = \tilde{X}_2$.

Proof. Let $\tilde{X}_1$ and $\tilde{X}_2$ are two generalized non linear fuzzy numbers such that $\tilde{X}_1 = \tilde{X}_2$. Therefore, $a_j = b_j$, $j = 1, 2, 3, 4$. So, $|a_j - b_j| = 0$. i.e., $\boxed{D_1(\tilde{X}_1, \tilde{X}_2) = 1}$.
Since $\tilde{X}_1 = \tilde{X}_2$, $a_j = b_j$, $j = 1, 2, 3, 4$, $w_1 = w_2$ and $q_1 = q_2$, so, $x_{i1} = x_{i2}$ and $y_{i1} = y_{i2}$, $i = 1, 2, 3$.
Therefore $\boxed{D_2(\tilde{X}_1, \tilde{X}_2) = 1}$.
Conversely, let $D_1(\tilde{X}_1, \tilde{X}_2) = 1$ and $D_2(\tilde{X}_1, \tilde{X}_2) = 1$.
Since, $D_1(\tilde{X}_1, \tilde{X}_2) = 1$ i.e., $|a_j - b_j| = 0$, i.e., $a_j = b_j$, $j = 1, 2, 3, 4$.
Again, $D_2(\tilde{X}_1, \tilde{X}_2) = 1$ i.e., $x_{i1} = x_{i2}$ and $y_{i1} = y_{i2}$, $i = 1, 2, 3$. Since, $a_j = b_j$ and $x_{i1} = x_{i2}$ so $\mu_{\tilde{X}_1}(x) = \mu_{\tilde{X}_2}(x)$. i.e., $w_1 = w_2$ and $q_1 = q_2$. Hence $\boxed{\tilde{X}_1 = \tilde{X}_2}$ □

Proposition 4. $S(\tilde{X}_1, \tilde{X}_2) \in [0,1]$.

Proof. As in Proposition 1, it is proved that $D_1(\tilde{X}_1, \tilde{X}_2) \in [0,1]$ and $D_2(\tilde{X}_1, \tilde{X}_2) \in [0,1]$. So it is obvious from equation (1) that $\boxed{S(\tilde{X}_1, \tilde{X}_2) \in [0,1]}$ □

Proposition 5. $S(\tilde{X}_1, \tilde{X}_2) = S(\tilde{X}_2, \tilde{X}_1)$.

Proof. From equation (1) we have

$$\begin{aligned} S(\tilde{X}_1, \tilde{X}_2) &= D_1(\tilde{X}_1, \tilde{X}_2) \times D_2(\tilde{X}_1, \tilde{X}_2) \\ &= D_1(\tilde{X}_2, \tilde{X}_1) \times D_2(\tilde{X}_2, \tilde{X}_1) \text{ from Proposition 2} \\ &= S(\tilde{X}_2, \tilde{X}_1) \end{aligned}$$

So, $\boxed{S(\tilde{X}_1, \tilde{X}_2) = S(\tilde{X}_2, \tilde{X}_1)}$ □

Proposition 6. $S(\tilde{X}_1, \tilde{X}_2) = 1$ *iff* $\tilde{X}_1 = \tilde{X}_2$.

Proof. From equation (1) we have

$$S(\tilde{X}_1, \tilde{X}_2) = D_1(\tilde{X}_1, \tilde{X}_2) \times D_2(\tilde{X}_1, \tilde{X}_2)$$

Now from Proposition 3, it is seen that $D_1(\tilde{X}_1, \tilde{X}_2) = 1$ and $D_2(\tilde{X}_1, \tilde{X}_2) = 1$ iff $\tilde{X}_1 = \tilde{X}_2$. So, it is easy to conclude that $\boxed{S(\tilde{X}_1, \tilde{X}_2) = 1 \text{ iff } \tilde{X}_1 = \tilde{X}_2}$. □

5. Examples

Here some examples are considered to check the efficiency of the proposed method.

Example 5.1.

Let $\tilde{X}_1 = (0.1, 0.2, 0.3, 0.4; 1, 1)$ and $\tilde{X}_2 = (0.1, 0.2, 0.3, 0.4; 1, 2)$ are two general fuzzy numbers. Graphically, these fuzzy numbers are shown in Figure 2. The ranges and heights of the fuzzy numbers are identical; however, their values of p are different. Therefore, the similarity between the two fuzzy numbers is not equal to 1. In the case of generalized trapezoidal fuzzy numbers, the similarity value would be equal to 1. However, using the proposed similarity technique, the obtained similarity value is $S(\tilde{X}_1, \tilde{X}_2) = 0.983806$. Hence, the proposed method satisfies the proposition of the similarity measure.

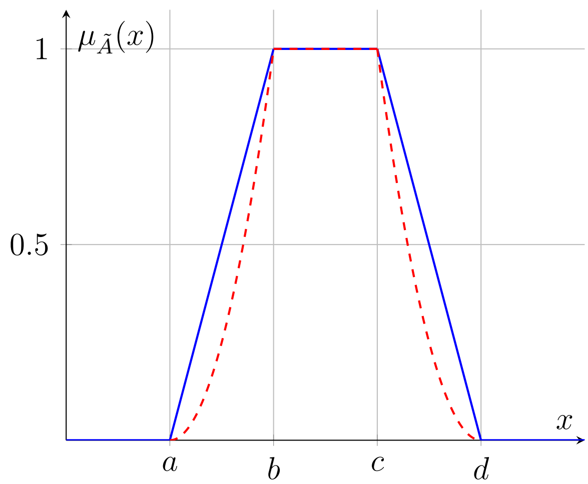


Fig. 2. Comparison of fuzzy membership functions with $\tilde{X}_1 = (0.1, 0.2, 0.3, 0.4; 1, 1)$ and $\tilde{X}_2 = (0.1, 0.2, 0.3, 0.4; 1, 2)$

Example 5.2.

Let $\tilde{X}_1 = (0.1, 0.2, 0.3, 0.4; 0.8, 2)$ and $\tilde{X}_2 = (0.1, 0.2, 0.3, 0.4; 1, 2)$ are two general fuzzy numbers. Graphically, these fuzzy numbers are shown in Figure 3. The ranges of the fuzzy numbers are identical, and the values of p are also the same; however, their heights are different. Therefore, the similarity between the two fuzzy numbers is not equal to 1. Using the proposed similarity technique, the obtained similarity value is $S(\tilde{X}_1, \tilde{X}_2) = 0.865836$, which satisfies the proposition of the similarity measure.

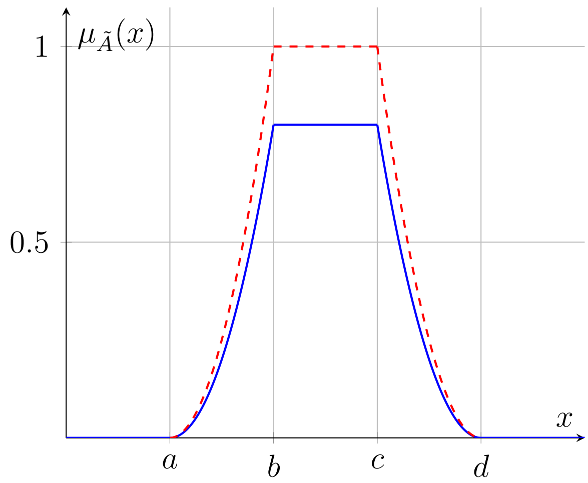


Fig. 3. Comparison of fuzzy membership functions with $\tilde{X}_1 = (0.1, 0.2, 0.3, 0.4; 0.8, 2)$ and $\tilde{X}_2 = (0.1, 0.2, 0.3, 0.4; 1, 2)$

Example 5.3.

Let $\tilde{X}_1 = (0.1, 0.2, 0.3, 0.4; 1, 2)$ and $\tilde{X}_2 = (0.1, 0.2, 0.3, 0.4; 1, 2)$ are two general fuzzy numbers. Graphically, these fuzzy numbers are shown in Figure 4. The ranges of the fuzzy numbers are identical, and both the height and the value of q are the same. Therefore, the similarity between the two fuzzy numbers is equal to 1. Using the proposed similarity technique, the obtained similarity value is $S(\tilde{X}_1, \tilde{X}_2) = 1$, which satisfies the proposition of the similarity measure.

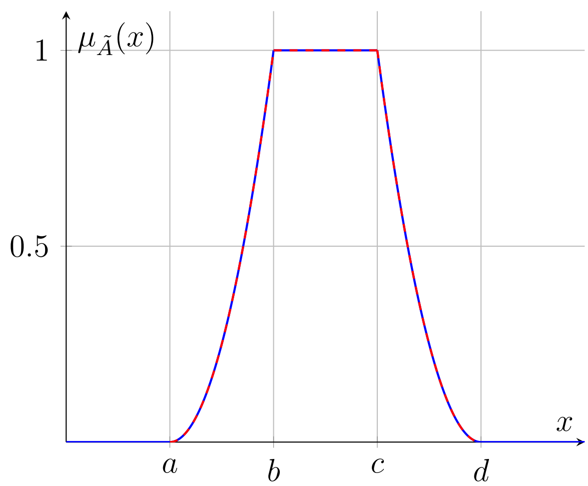


Fig. 4. Comparison of fuzzy membership functions with $\tilde{X}_1 = (0.1, 0.2, 0.3, 0.4; 1, 1)$ and $\tilde{X}_2 = (0.1, 0.2, 0.3, 0.4; 1, 1)$

Therefore, in all different cases, the similarity value can be easily calculated by the proposed technique.

6. Comparison Results with Existing Methods

Although no similarity measure technique exists for generalized non linear trapezoidal fuzzy numbers, a comparison with 12 different examples, which are given in Table 1, has been made to show the efficiency of the proposed method. The comparison has been made with some existing techniques for generalized trapezoidal fuzzy numbers. From the discussion, it can be shown that the proposed similarity measure is able to determine the similarity value of any kind of fuzzy number given in linear or non linear form, whereas the existing methods are unable to perform the exact measure.

Table 1: Tabular representation of the examples

	$\tilde{X}_1$	$\tilde{X}_2$
Example-6.1	(0.1,0.2,0.3,0.4;1,2)	(0.1,0.25,0.25,0.4;1,1)
Example-6.2	(0.1,0.2,0.3,0.4;1,1)	(0.1,0.25,0.25,0.4;1,1)
Example-6.3	(0.1,0.2,0.3,0.4;1,1)	(0.5,0.65,0.65,0.8;1,1)
Example-6.4	(0.1,0.2,0.3,0.4;1,2)	(0.5,0.65,0.65,0.8;1,1)
Example-6.5	(0.1,0.2,0.3,0.4;1,2)	(0.3,0.45,0.45,0.6;1,1)
Example-6.6	(0.1,0.2,0.3,0.4;1,1)	(0.3,0.45,0.45,0.6;1,1)
Example-6.7	(0.1,0.2,0.3,0.4;1,1)	(0.1,0.25,0.25,0.4;0.2,1)
Example-6.8	(0.1,0.2,0.3,0.4;1,2)	(0.1,0.25,0.25,0.4;0.2,1)
Example-6.9	(0.1,0.2,0.3,0.4;1,1)	(0.1,0.35,0.35,0.5;1,1)
Example-6.10	(0.1,0.2,0.3,0.4;1,2)	(0.1,0.35,0.35,0.5;1,1)
Example-6.11	(0.1,0.2,0.3,0.4;1,1)	(0.1,0.2,0.3,0.4;1,1)
Example-6.12	(0.1,0.2,0.3,0.4;1,2)	(0.1,0.2,0.3,0.4;1,1)

The similarity measure approach proposed by different authors is calculated for the examples as given in Table 1. All the results are shown in Table 2.

Table 2: Tabular representation of comparison results

Similarity value of	Wei and Chen [3]	Xu et al [4]	Hejazi et al. [5]	Patra and Mondal [7]	Dutta and Borah [12]	Proposed Method
Example-6.1	0.9499	0.9627	0.9004	0.9506	0.998073	0.91838
Example-6.2	0.9499	0.9627	0.9004	0.9506	0.998073	0.95968
Example-6.3	0.5846	0.6194	0.5555	0.585	0.84019	0.503555
Example-6.4	0.5846	0.6194	0.5555	0.585	0.84019	0.460934
Example-6.5	0.7794	0.8072	0.7407	0.78	0.947702	0.762331
Example-6.6	0.7794	0.8072	0.7407	0.78	0.947702	0.706773
Example-6.7	0.2859	0.8434	0.0644	0.5021	0.605505	0.885756
Example-6.8	0.2859	0.8434	0.0644	0.5021	0.605505	0.814383
Example-6.9	0.1583	0.7704	0	0.36	0.987522	0.876793
Example-6.10	0.1583	0.7704	0	0.36	0.987522	0.875042
Example-6.11	1	1	1	1	1	1
Example-6.12	1	1	1	1	1	0.983823

From Table 2, it can be observed that the existing methods fail to differentiate the similarity between linear and nonlinear generalized trapezoidal fuzzy numbers. Considering Examples 6.1 and 6.2, it is clear that the similarity values obtained by the existing methods are identical. However, from Table 1, it can be easily seen that one of the fuzzy numbers remains the same, whereas the other differs in

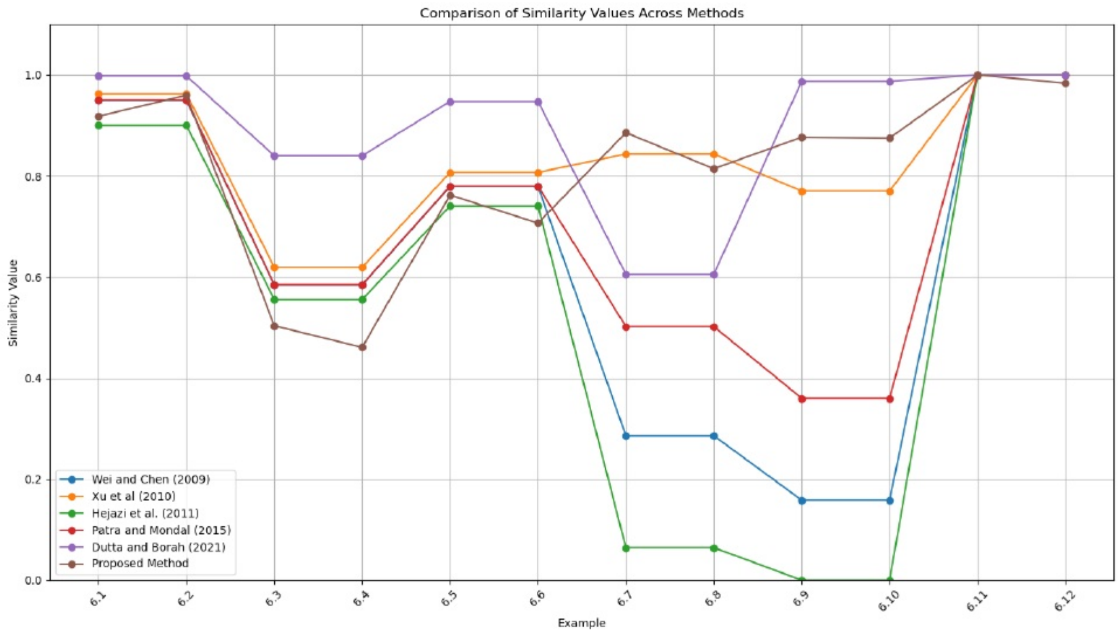


Fig. 5. Comparison results with other methods

the two examples. Therefore, the similarity values should not be identical. This drawback is effectively addressed by the proposed method, as shown in Table 2. Similar limitations can also be observed in Examples 6.3 and 6.4, Examples 6.5 and 6.6, Examples 6.7 and 6.8, Examples 6.9 and 6.10, and Examples 6.11 and 6.12. These shortcomings are successfully overcome by the proposed technique. Therefore, the proposed method is highly suitable for calculating the similarity of generalized nonlinear trapezoidal fuzzy numbers as well as generalized linear trapezoidal fuzzy numbers. The comparison results are also graphically illustrated in Figure 5.

7. Application of the Proposed Method in a Multi-Attribute Decision-Making Problem

Let there are m alternatives namely $A_1, A_2, ..., A_m$ and also let n attributes are there namely $C_1, C_2, ..., C_n$. It is considered that for the alternatives A_i, the decision maker assesses the attribute C_j, which are taken in the form of generalized nonlinear trapezoidal fuzzy numbers $\tilde{d}_{ij}$, $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$. So we get a decision matrix $\tilde{D} = (\tilde{d}_{ij})_{m\times n}$ whose elements $\tilde{d}_{ij}$ are the generalized nonlinear trapezoidal fuzzy number of the form $\tilde{d}_{ij} = (d_{ij1}, d_{ij2}, d_{ij3}, d_{ij4}; w_{ij}, p_{ij})$. i.e.,

$$\tilde{D} = \begin{matrix} & \\ A_1 \\ A_2 \\ . \\ . \\ A_m \end{matrix} \begin{matrix} \begin{matrix} C_1 & C_2 & ... & C_n \end{matrix} \\ \begin{pmatrix} \tilde{d}_{11} & \tilde{d}_{12} & ... & \tilde{d}_{1n} \\ \tilde{d}_{21} & \tilde{d}_{22} & ... & \tilde{d}_{2n} \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ \tilde{d}_{m1} & \tilde{d}_{m2} & ... & \tilde{d}_{mn} \end{pmatrix} \end{matrix}$$

where $1 \le i \le m, 1 \le j \le n$.

Let a decision maker have some weight for each attribute w_j, $1 \le j \le n$ such that $\sum_{j=1}^{n} w_j = 1$. Now one decision maker has to choose the best alternative. The selection can be done as follows:

Step-1: First, consider an ideal solution for the decision maker in the form of a generalized nonlinear trapezoidal fuzzy number, say $\tilde{IS} = (I_1, I_2, I_3, I_4; w_{IS}, p_{IS})$.
Step-2: Then calculate the similarity between $\tilde{IS}$ and each $\tilde{d}_{ij}$ of the decision matrix $\tilde{D}$. So, a new crisp matrix $M = (s_{ij})_{m\times n}) = (S(\tilde{IS}, \tilde{d}_{ij}))_{m\times n}$ can be determine.
Step-3: In the decision matrix, some of the criteria may be preferable when they attain their maximum values, while some criteria may be preferable when they attain their minimum values. Therefore, to obtain the best alternative, the problem has to be converted either into maximization form or minimization form by transforming the elements of the matrix $M = (s_{ij})$ into $M' = (s'_{ij})$. When converting maximization criteria into minimization criteria, the following transformation is used:

$$s'_{ij} = \begin{cases} s_{ij}, & \text{if it is in minimize form} \\ 1 - s_{ij} & \text{if it is in maximize form} \end{cases} \tag{2}$$

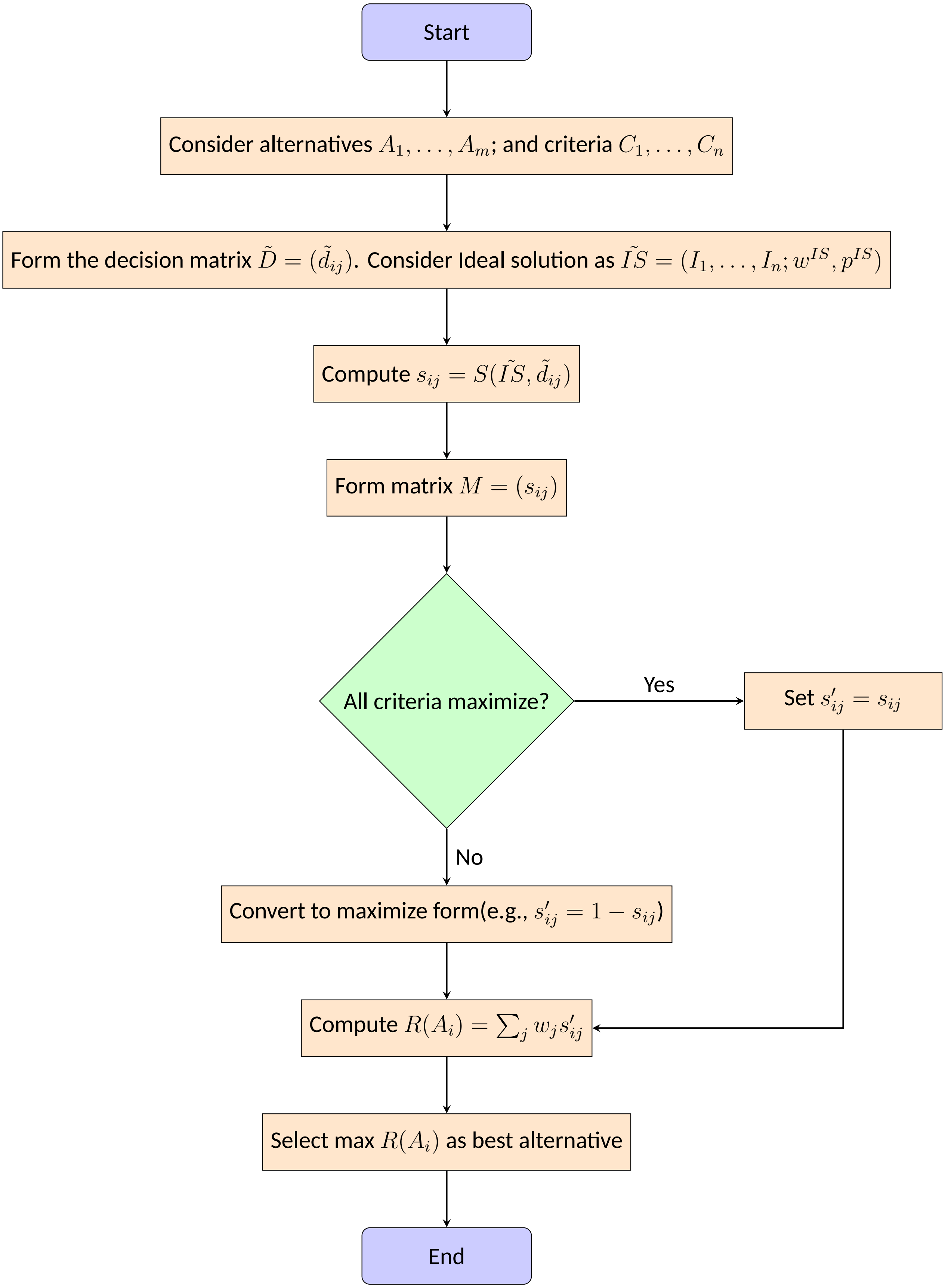
If we want the matrix in maximized form, we can calculate

$$s'_{ij} = \begin{cases} s_{ij}, & \text{if it is in maximize form} \\ 1 - s_{ij} & \text{if it is in minimize form} \end{cases} \tag{3}$$

Step-4: Now, for each alternative A_i ($i = 1, 2, ..., m$), determine the rank value of each alternative by

$$R(A_i) = \sum_{j=1}^{n} w_j.s'_{ij}$$

Step-5: The maximum value of $R(A_i)$ will be the best choice for the decision maker if he/she want it in maximize form and The minimum value of $R(A_i)$ will be the best choice for the decision maker if he/she want it in minimize form.



To show the applicability of the method, the following example has been considered

Example 7.1.

Let a builder want to start a project. For this purpose, he/she considers four locations, namely A_1,

A_2, A_3, and A_4. To undertake the project, many factors are associated with the selection of a suitable location. Among these factors, four major criteria are considered: Land Cost (C_1), Transportation Cost (C_2), Quality of Land (C_3), and Availability of Labour (C_4). The information related to these criteria is, in most cases, imprecise in nature. Therefore, the corresponding criterion values are represented in the form of generalized trapezoidal fuzzy numbers, and the corresponding values are given in the following decision matrix

$$\tilde{D} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{pmatrix} (0.5, 0.65, 0.65, 0.8; 1, 1) & (0.7, 0.8, 0.8, 0.9; 1, 2) & (0.8, 0.8, 0.8, 0.8; 1, 1) & (0.6, 0.7, 0.8, 0.9; 1, 2) \\ (0.2, 0.3, 0.4, 0.5; 1, 1) & (0.8, 0.85, 0.85, 0.9; 1, 1) & (0.6, 0.7, 0.7, 0.8; 1, 2) & (0.6, 0.7, 0.8, 0.9; 1, 2) \\ (0.3, 0.4, 0.4, 0.5; 1, 1) & (0.6, 0.7, 0.7, 0.8; 1, 2) & (0.7, 0.8, 0.8, 0.9; 1, 2) & (0.5, 0.6, 0.7, 0.8; 1, 1) \\ (0.4, 0.4, 0.4, 0.4; 1, 1) & (0.8, 0.85, 0.85, 0.9; 1, 1) & (0.6, 0.7, 0.7, 0.8; 1, 2) & (0.8, 0.8, 0.8, 0.8; 1, 1) \end{pmatrix} \end{array}$$

Now, out of these places A_1, A_2, A_3, A_4, the builder has to choose the best suitable place for the project with minimum cost.
Let the ideal solution for the decision maker is $IS = (1, 1, 1, 1; 1, 1)$.
So, we get the similarity value matrix of the ideal solution with each element of the decision matrix as follows:

$$M = \begin{pmatrix} 0.508244 & 0.678045 & 0.707624 & 0.61818 \\ 0.215103 & 0.750424 & 0.561877 & 0.61818 \\ 0.386667 & 0.561877 & 0.678045 & 0.506779 \\ 0.261436 & 0.750424 & 0.561877 & 0.707624 \end{pmatrix}$$

Now, the builder will always want to minimize his/her costs, i.e., C_1 and C_2 are in minimization form. However, land quality and availability of labour need to be maximized. Therefore, to solve the problem, the values of C_3 and C_4 are converted from maximization form to minimization form using the formula $s'ij = 1 - sij$. Hence, the decision table is converted to

$$M' = \begin{pmatrix} 0.508244 & 0.678045 & 0.292376 & 0.38182 \\ 0.215103 & 0.750424 & 0.438123 & 0.38182 \\ 0.386667 & 0.561877 & 0.321955 & 0.493221 \\ 0.261436 & 0.750424 & 0.438123 & 0.292376 \end{pmatrix}$$

Let the decision maker have the weight matrix for the attributes as $(0.2, 0.25, 0.25, 0.3)$.

Now using the proposed algorithm we can determine the rank value of the alternative as $R(A_1) = 0.4588, R(A_2) = 0.4547, R(A_3) = 0.4463$ and $R(A_4) = 0.4371$. Hence we get $R(A_1) > R(A_2) > R(A_3) > R(A_4)$. Now, the builder wants the minimum rank value. So, for the given problem, A_4 is the best alternative for the decision maker.

8. Conclusion

In this present work, a new similarity measure technique has been introduced for generalized non linear trapezoidal fuzzy numbers. No previous work has been done by any researchers for this type of fuzzy number. Therefore, for the first time, we have introduced such a method. The proposed method is based on the distance measure and COG measure of the given fuzzy numbers. Some propositions have been derived from the proposed method. Also, some examples have been considered to compare the method with the existing similarity measure techniques for generalized trapezoidal fuzzy numbers of linear form. From the comparison, it is clear that the existing methods give the same similarity value for generalized trapezoidal fuzzy numbers and generalized non linear trapezoidal fuzzy numbers, whereas the proposed technique can overcome this drawback. Also, the proposed technique can be applied to find the similarity value of both types of fuzzy numbers, i.e., generalized trapezoidal fuzzy numbers and generalized non linear trapezoidal fuzzy numbers. Finally, an algorithm has been introduced for solving a multi attribute decision making problem using the proposed similarity measure technique, and it is seen that the method is very much suitable for solving MADM problems.

Future Research Scope

Here, the similarity measure has been proposed only for non linear generalized trapezoidal fuzzy numbers, but it can be extended to nonlinear interval-valued trapezoidal fuzzy numbers or non linear intuitionistic fuzzy numbers. Also, the method is applied to a project selection model. However, the similarity measure technique can also be applied to other decision-making problems, such as risk analysis and fault diagnosis problems.

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Conflicts of Interest
The author declares no conflicts of interest.

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