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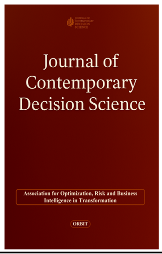

Business-oriented Stock Market Decision Analysis Using Circular Complex Picture Fuzzy Sets and Advanced MCDM Based on the CRITIC–WASPAS Method

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ABSTRACT

Multi-criteria evaluation and financial sustainability analysis in stock markets often involve uncertain and imprecise information, which requires advanced decision-making models. In this paper, we discuss the concepts of Circular Complex Picture Fuzzy Sets (CrC-PiFS) for handling uncertainty in financial assessments. The circular complex T-spherical fuzzy set is an extension of the complex picture fuzzy set, complex spherical fuzzy set, and complex T-spherical fuzzy set. We define improved algebraic operations for CrC-PiFS, including direct sum, direct product, and scalar multiplication, based on t-norms and t-conorms. The aim of this study is to enhance the representation of uncertainty in multi-criteria stock-market decision-making. To achieve this, we introduce circular complex picture fuzzy weighted/ordered weighted arithmetic mean and geometric mean aggregation operators under a new class of algebraic circular complex T-spherical fuzzy operational laws, and discuss their properties. Finally, we present a novel decision-making framework incorporating the CRITIC–WASPAS method and highlight its applicability to stock-market analysis, particularly in evaluating and prioritizing market stability factors, investment strategies, and financial sustainability indicators.

1. Introduction

1.1 *Short review of T-spherical fuzzy set*

The concept of fuzzy set (FS) was initiated by Zadeh in 1965 [1] to help manage the modeling of real-world issues involving ambiguous data. FS theory is particularly a helpful tool for modeling

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ambiguity; it may be used to model and solve problems in a wide range of fields, including data mining, grouping, and medical research. An FS is defined by a membership function (MF) η from a collection of the universe's objects or elements to the interval $[0,1]$, where the MF η belongs to the range $[0,1]$. Subsequently, Atanassov [2] extended FS by adding a non-membership function (NMF) ψ to present the idea of intuitionistic fuzzy sets (IFS), ensuring the addition of MF and NMF satisfies $0 \leq \eta + \psi \leq 1$. However, in cases involving the pairs like $(0.7, 0.6)$, the concept of IFS did not satisfy this condition, that is, $0.7 + 0.6 = 1.3 > 1$. Therefore, Yager [3] introduced the Pythagorean fuzzy set $(PyFS)$ by applying the new condition $0 \leq \eta^2 + \psi^2 \leq 1$. However, the case $(0.8, 0.9)$ did not satisfy the condition of $PyFS$, that is, $0.8^2 + 0.9^2 > 1$. In order to remove this kind of restriction, Yager [4] developed the concept called q-rung orthopair fuzzy sets $(q$-ROFS$)$, with the restriction $0 \leq \eta^q + \psi^q \leq 1$, where $q \geq 1$, which covers the failure situations.

Real-world decision-making (DM) can produce additional responses for a choice index, such as neutral, abstention, negative, and positive. For this, Cuong [5] initiated the concept of picture fuzzy sets $(PiFS)$, which handle the membership value (MV) η, non-membership value (NMV) ψ, and neutral value (NV) ϕ, with the condition $0 \leq \eta + \psi + \phi \leq 1$. However, sometimes it is challenging to accept certain restrictions; for example, $0.2 + 0.8 + 0.1 = 1.1 > 1$ does not belong to $[0,1]$. The spherical fuzzy set (SFS) was introduced by Mahmood *et al.* [6], which is more effective compared to $PiFS$ and IFS, with the condition that the sum of squares of MV, NMV, and NV is less than or equal to 1, that is, $0 \leq \eta^2 + \psi^2 + \phi^2 \leq 1$. However, if the values are $(0.7, 0.6, 0.9)$, this condition is not satisfied since $0.7^2 + 0.6^2 + 0.9^2 = 1.66 > 1$. To handle such situations, Ullah *et al.* [7] initiated the T-spherical fuzzy set $(TSFS)$ as an extension of SFS, with the condition $0 \leq \eta^q + \psi^q + \phi^q \leq 1$, where $q > 1$. These developments provide useful tools for simulation, decision-making, environmental sustainability, and the advancement of theoretical and real-world implementations of intelligent systems.

1.2 *Short review of complex t-spherical fuzzy set*

In real-world settings, decision-makers (DMs) face more obstacles in selecting the best option from a variety of workable possibilities as systems become more complicated. While this presents a significant challenge, achieving a single goal is not insurmountable. Many organizations struggle with the complexities of setting goals, forming viewpoints, and motivating employees. Decisions made by committees or individuals within an organization therefore take into account several simultaneous objectives. Each DM is compelled to develop the best solution for realistic implications in real-world problems by using criteria that allow alternative responses. As a result, DMs are becoming more committed to creating dependable and practical techniques for identifying optimal solutions. As discussed earlier, existing methods have limitations and are unable to fully capture uncertainty and its variations over time. Consequently, researchers raised the question of what would occur if the range of fuzzy sets (FSs) were extended to a unit disk in the complex plane. Accordingly, Ramot *et al.* [8] initiated the concept of complex fuzzy sets (CFS). The fundamental idea of CFS is to extend the membership range from $[0,1]$ to the complex plane. The membership function is defined as $\eta = re^{i\theta}$, where r is the amplitude term belonging to $[0,1]$ and θ is the phase (or periodic) term belonging to $[0, 2\pi]$. The CFS framework considers only the membership function; however, in many situations, using only a complex-valued membership function $(CVMF)$ is insufficient. To address this issue, Alkouri and Salleh [9] introduced the complex intuitionistic fuzzy set $(CIFS)$ by incorporating a complex-valued non-membership function $(CVNMF)$. A $CIFS$ is defined by the $CVMF$ $\eta_c = r_c e^{iw_{\eta_c}}$ and $CVNMF$ $\psi_c = k_c e^{iw_{\psi_c}}$, where r_c, k_c, w_{η_c}, and w_{ψ_c} belong to $[0,1]$. The restriction of $CIFS$ is given by $|\eta_c + \psi_c| \leq 1$. However, due to the presence of only two membership components, certain complex fuzzy data remain difficult to interpret. For instance, if the decision-maker assigns $0.5e^{2\pi i(0.42)}$ as the membership value and $0.7e^{2\pi i(0.79)}$ as the non-membership value, the $CIFS$ condition is violated. To overcome such limitations, Ullah *et al.* [10] developed the complex Pythagorean fuzzy set $(CPyFS)$, in which the sum of squares of the $CVMF$ and $CVNMF$ is required to be less than or equal to 1. While $CPyFS$ is more flexible than $CIFS$, it still fails in certain cases. For example, when the complex-valued membership degree is $0.8e^{2\pi i(0.92)}$ and the complex-valued non-membership degree is $0.7e^{2\pi i(0.81)}$, both $CIFS$ and $CPyFS$ violate their respective conditions. To handle such scenarios, Liu *et al.* [11] introduced the complex q-rung orthopair fuzzy set (Cq-ROFS), in which the sum of the qth powers of the $CVMF$ and $CVNMF$ is less than or equal to 1. The proposed Cq-ROFS provides an effective framework for managing ambiguous and complex information and has been successfully applied to multi-attribute decision making $(MADM)$ problems. However, Cq-ROFS cannot model situations involving a complex neutral function (CNF), as it only considers the $CVMF$ and $CVNMF$. To address this limitation, Akram *et al.* [12] introduced the complex picture fuzzy set $(CPiFS)$, where the amplitude terms $(\eta, \psi, \phi) \in [0, 1]$ and the phase terms $(\theta, \delta, \gamma) \in [0, 2\pi]$, subject to defined constraints. Nevertheless, certain amplitude combinations violate these conditions. Furthermore, Akram et al. [13] proposed the complex spherical fuzzy set $(CSFS)$ as a generalization of $CPiFS$. Later, Ali *et al.* [14] introduced the complex T-spherical fuzzy set (CT-SFS), which successfully addresses the failure cases encountered in previous complex fuzzy extensions.

1.3 *Short review of cricular q-rung orthopair fuzzy set*

Another question raised by many researchers is what would happen if we transform the range of intuitionistic fuzzy sets $(IFSs)$ into a circular region. Therefore, Atanassov [15] initiated the concept of circular intuitionistic fuzzy sets $(CIFSs)$. Furthermore, Bozyigit *et al.* [16] developed the circular Pythagorean fuzzy set (Cr-PFS) as an extension of Cr-IFS, where Cr-PFS contains the membership degree η and the non-membership degree ψ under the condition $0 \leq \eta^2 + \psi^2 \leq 1$, with a radius $R \in [0, 1]$ of a circle centered at the point (η, ψ) in the plane. Recently, Yusoff *et al.* [17] extended Cr-PFS to circular q-rung orthopair fuzzy sets ($Cirq$-ROFSs), which satisfy the condition $0 \leq \eta^q + \psi^q \leq 1$, $q \geq 1$, with a radius $R \in [0, 1]$ of a circle around the point (η, ψ), and established their basic algebraic properties. However, Zeeshan *et al.* [18] identified novel characteristics of $Cirq$-ROFSs, including flexible algebraic laws and Dombi aggregation laws. Furthermore, Kahraman and Otay [19] employed the VIKOR method using circular intuitionistic fuzzy sets to address decision-making problems.

1.4 *The main motivations*

In general, the complex T-spherical fuzzy set (CT-SFS) is a wide generalization of fuzzy sets. However, CT-SFS cannot handle well enough for some circumstances. In this paper, we propose the novel approach of Cricular Complex Picture Fuzzy Set (CrC-PiFS) with its operational laws where CrC-PiFS composes the grade of membership, abstinence, and non-membership with a condition in which the sum of q-power of the real part (also for imaginary part) of the membership, abstinence, and non-membership grades is not exceeded from a unit interval.

Multi-criteria decision-making (MCDM) techniques are becoming more and more popular as prospective tools for assessing and resolving complicated real-time issues because of their innate capacity to compare numerous options based on a range of factors in order to potentially choose the optimal option. MCDM challenges include various uniqueness such as presence of multiple non-commensurable and conflicting criteria, distinct units of measurement among the criteria, also presence of relatively dissimilar alternatives. There are several MCDM techniques being used to handle these decision-making issues that describe multidimensional scenarios. The MCDM methods are primarily targeted at analyzing and ranking the available alternatives. There are numerous cases when various MCDM techniques produce disparate outcomes (i.e., the ranks of the same alternatives vary according on the techniques used). This can be attributable to the many mathematical artifacts used by the methodologies under consideration. The WASPAS method is a unique combination of two well-known MCDM approaches, i.e. weighted sum model (WSM) and weighted product model (WPM). Its application first requires development of a decision/evaluation matrix, $X = [x_{ij}]_{m\times n}$ where x_{ij} is the performance of i^{th} alternative with respect to j^{th} criterion, m is the number of alternatives and n is the number of criteria. The feasible alternatives are now ranked based on the Q values and the best alternative has the highest Q value. Table 1 outlines the specific limitations of earlier research in contrast to the proposed approach.

Table 1 Comparison of proposed model with extant models in literature.

Concept	MD	NMD	ND	CMD	CNMD	CND	Radius	λ	β
FS	Yes	No	No	No	No	No	No	No	No
IFS	Yes	Yes	Yes	No	No	No	No	No	No
PyFS	Yes	Yes	Yes	No	No	No	No	No	No
q-ROFS	Yes	Yes	Yes	No	No	No	No	No	No
PiFS	Yes	Yes	Yes	No	No	No	No	No	No
SFS	Yes	Yes	Yes	No	No	No	No	No	No
T-SFS	Yes	Yes	Yes	No	No	No	No	No	No
CrIFS	Yes	Yes	Yes	No	No	No	Yes	No	No
CrPyFS	Yes	Yes	Yes	No	No	No	Yes	No	No
Crq-ROFS	Yes	Yes	Yes	No	No	No	Yes	No	No
CFS	Yes	No	No	Yes	No	No	No	No	No
CIFS	Yes	Yes	Yes	Yes	Yes	Yes	No	No	No
CPyFS	Yes	Yes	Yes	Yes	Yes	Yes	No	No	No
Cq-ROFS	Yes	Yes	Yes	Yes	Yes	Yes	No	No	No
CPiFS	Yes	Yes	Yes	Yes	Yes	Yes	No	No	No
CSFS	Yes	Yes	Yes	Yes	Yes	Yes	No	No	No
CT-SFS	Yes	Yes	Yes	Yes	Yes	Yes	No	No	No
(Proposed CrC-PiFS)	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes

We found from the aforementioned investigation that the following are the main issues that all experts have:

(i) On the basis of CrC-PiFSs, how should novel operational laws be drafted?

(ii) On the basis of novel operational laws, how may new operators be developed?

(iii) How can all actions be ranked according to the developed operators?

Table 2
Related work circular complex picture fuzzy set

Author and year	Benchmarks	Application
Abra Hussain *et al.* [20]	Decision-Making based on Complex Picture Fuzzy	Criteria for Supplier Selection
M. Shoaib *et al.* [21]	Certain Operations on Complex Picture Fuzzy Graphs	Graphs
P. Liu *et al.* [22]	Decision making based on complex picture fuzzy knowledge	Solar Panels
H. Dhumras *et al.* [23]	Similarity Measures of Complex Picture Fuzzy	Medical Diagnosis
Zaid Khan *et al.* [24]	Problems under complex picture fuzzy sets	Medical Diagnosis
S. U. Khan [25]	Concept of Complex Picture Fuzzy Soft Information	Network
M. N. Khan *et al.* [26]	Decision-making model complex cubic picture fuzzy information	Enterprise resource planning
Zeeshan Ali *et al.* [27]	Power Aggregation Operators for Complex Picture Fuzzy	Marketing

1.5 *Novelty and main contributions*

This study aims to create a logical and analytical approach for decision support so that the best option can be chosen from a variety of alternative sources. The incorporation of algebraic complex operational rules into the CrC-PiFS environment will enable the use of the $CrCT$-SFS arithmetic and geometric mean aggregation operators, which will guarantee the efficiency of the conceptual framework.

The following highlights the main goals and contributions of this article:

(i) Being a modified version of T-SFS and CT-SFS is more capable, comprehensive, and dependable than current concepts like CrC-PiFS when it comes to handling uncertain data during the decision-making process. Furthermore, the coupling of CrC-PiFS circumstances with arithmetic and geometric mean aggregation operators using algebraic complex operational laws have not been studied before. Therefore, enhancing the geometric and arithmetic mean aggregation operators utilizing complex operational principles is crucial for solving MCDM problems in CrC-PiFS situations.

(ii) An insightful idea for handling three-dimensional data in a single set is the arithmetic and geometric mean aggregation operators pursuant to CrC-PiFS settings, which employ complex operational rules. Therefore, the purpose of this study is to offer Crcular Complex-PiFS weighted arithmetic mean aggregation operator (CrC-PiFWAM), Crcular Complex T-SFS ordered weighted arithmetic mean aggregation operator (CrC-PiSFOWAM), Crcular Complex PiSFS weighted geometric mean aggregation operator (CrC-PiFWGM) and Crcular Complex -PiFS ordered weighted geometric mean aggregation operator (CrC-PiFOWGM).

(iii) The connections between these operators are highlighted in order to address certain of their characteristics, including their boundedness, idempotency, and monotonicity.

(iv) To create two distinct, cutting-edge methods based on the CrC-PiFWAM and CrC-PiSFWGM.

(v) We give a graphical depiction of the proposed approach to further enhance the clarity and understandability of the created procedure. Using a flowchart to graphically depict the intended process, the recommended modelling technique is clearly communicated.

(vi) To provide application that demonstrate the viability and dependability of the suggested techniques. Additionally, by contrasting the suggested techniques with the current approaches, we will demonstrate that the suggested techniques are better and that the procedure for aggregation is more adaptable when using the arithmetic and geometric mean aggregation operators in accordance with CrC-PiFS environments and complex operational rules.

CrC-PiFS arithmetic and geometric mean aggregation operators based on algebraic complex operational rules offer a high degree of flexibility when aggregating confusing data. By adjusting the parameters, one can change the focus placed on specific fuzzy MD, NMD, ND and their related the parameter. These operators have been effectively applied in domains such as expert systems, data fusion, pattern recognition, and decision-making. Their ability to handle complicated and unpredictable information makes them suitable for managing ambiguous and inaccurate real-world situations. Their unique approach to data aggregation yields findings that are more reliable and accurate, assisting decision-makers in a range of applications in making better decisions.

1.6 The structure of the paper

According to the following, this article is organized: The main topic of section 2 is the construct of complex operational laws based Circular Complex Picture fuzzy weighted arithmetic mean aggregation operator (CrC-PiFWAM), Circular Complex Picture fuzzy ordered weighted arithmetic mean aggregation operator (CrC-PiFOWAM). The concept of Circular Complex Picture fuzzy weighted geometric mean aggregation operator (CrC-PiFWGM) and Circular Complex Picture fuzzy ordered weighted geometric mean aggregation operator (CrC-PiFOWGM) within the context of Circular Complex Picture fuzzy sets and their characteristics are covered in Section 3. Section 4 suggests a novel approach of decision-making using the cutting-edge methods based on the CrC-PiFWAM and CrC-PiFWGM. Additionally, Section 5 provides an example to demonstrate the value of the suggested strategy for making decisions. Section 6 characterizes the comparability and sensitivity analysis that demonstrate the logic and stability of the suggested technique. Section 7 provides a conclusion to the article.

2. Circular complex picture fuzzy weighted arithmetic mean aggregation operators

We provide definitions of Circular Complex Picture fuzzy operational laws for CrC-PiFNs in the following section. Following them, several aggregation (circular complex Picture fuzzy weighted arithmetic mean aggregation operator (CrC-PiFWAM), circular complex PiSF ordered weighted arithmetic mean aggregation operator (CrC-PiFOWAM)) operators based on circular complex picture fuzzy operational laws will be created.

[**ramot2002**] A CFS A is defined as:

$$A = \{(x, \eta(x)) \mid x \in X\}$$

where $\eta(x) = \eta(x)e^{i2\pi\omega(x)}$ denotes the grade of complex-valued truth with the condition $0 \leq \eta(x), \omega(x) \leq 1$.

[**akram2020a**] Let X be a universal set. A *complex picture fuzzy set* (CPFS) $\widetilde{P}$ on X is defined as

$$\widetilde{P} = \left\{ \left(x,\, \mu_{\widetilde{P}}(x),\, \eta_{\widetilde{P}}(x),\, \nu_{\widetilde{P}}(x)\right) : x \in X \right\},$$

where

$$\mu_{\widetilde{P}}(x) = \mu_r(x) e^{i\theta_\mu(x)}, \qquad \eta_{\widetilde{P}}(x) = \eta_r(x) e^{i\theta_\eta(x)}, \qquad \nu_{\widetilde{P}}(x) = \nu_r(x) e^{i\theta_\nu(x)},$$

denote the complex positive, neutral, and negative membership degrees of x, respectively. The magnitudes satisfy

$$0 \le \mu_r(x),\, \eta_r(x),\, \nu_r(x) \le 1, \qquad \mu_r(x) + \eta_r(x) + \nu_r(x) \le 1.$$

The complex refusal degree associated with x is given by

$$\pi_{\widetilde{P}}(x) = \left(1 - \mu_r(x) - \eta_r(x) - \nu_r(x)\right) e^{i\theta_\pi(x)}.$$

2.1 *Proposed circular complex picture fuzzy sets*

One aim of this study is to explore the novel approach of CrC-PiFSs and their operational laws. These operational laws are also verified with the help of a numerical example.

A CrC-PiFS P is defined as:

$$P = \{(x, \eta(x), \phi(x), \psi(x), r(x)) \mid x \in X\}$$

where $\eta(x) = \eta_{C_1} e^{i2\pi\eta_{C_1}^{im}}$, $\phi(x) = \phi_{C_1} e^{i2\pi\phi_{C_1}^{im}}$, $\psi(x) = \psi_{C_1} e^{i2\pi\psi_{C_1}^{im}}$, and $r(x) = r_{C_1} e^{i2\pi r_{C_1}^{im}}$denote the membership degree, abstinence, and non-membership with the conditions: $0 \le \eta_{C_1} + \phi_{C_1} + \psi_{C_1} \le 1$ and $0 \le (\eta_{C_1}^i + \phi_{C_1}^i + \psi_{C_1}^i) \le 1$. Additionally, the term $H(x) = Re^{i2\pi\omega_R(x)}$ such that $R = (1 - \eta_{C_1} + \phi_{C_1} + \psi_{C_1})$ and $\omega_R(x) = \left(1 - (\eta_{C_1}^i + \phi_{C_1}^i + \psi_{C_1}^i)\right)$ expresses the complex hesitancy grade of x. Moreover, $P = \left(\eta_{C_1} e^{i2\pi\eta_{C_1}^{im}}, \phi_{C_1} e^{i2\pi\phi_{C_1}^{im}}, \psi_{C_1} e^{i2\pi\psi_{C_1}^{im}}, r_{C_1} e^{i2\pi r_{C_1}^{im}}\right)$ is called a CrC-PiFN.

For any CrC-PiFN. $P_1 = \left(\eta_{C_1} e^{i2\pi\eta_{C_1}^{im}}, \phi_{C_1} e^{i2\pi\phi_{C_1}^{im}}, \psi_{C_1} e^{i2\pi\psi_{C_1}^{im}}, r_{C_1} e^{i2\pi r_{C_1}^{im}}\right)$, the score and accuracy functions are defined by

$$SC\left(P_1\right) = \frac{1}{8} \left\{ (\eta_{C_1}) + \left(\eta_{C_1}^{im}\right) + (r_{C_1}) + \left(r_{C_1}^{im}\right) - (\psi_{C_1}) - \left(\psi_{C_1}^{im}\right) - (\phi_{C_1}) - \left(\phi_{C_1}^{im}\right) \right\}$$

and

$$AC\left(P_1\right) = \frac{1}{8} \left\{ (\eta_{C_1}) + \left(\eta_{C_1}^{im}\right) + (\psi_{C_1}) + \left(\psi_{C_1}^{im}\right) + (\phi_{C_1}) + \left(\phi_{C_1}^{im}\right) + (r_{C_1}) + \left(r_{C_1}^{im}\right) \right\}$$

where $SC\left(P_1\right) \in [-1, 1]$ and $AC\left(P_1\right) \in [0, 1]$.

Let $P_1 = \left(\eta_{C_1} e^{i2\pi\eta_{C_1}^{im}}, \phi_{C_1} e^{i2\pi\phi_{C_1}^{im}}, \psi_{C_1} e^{i2\pi\psi_{C_1}^{im}}, r_{C_1} e^{i2\pi r_{C_1}^{im},}\right)$,

$P_2 = \left(\eta_{C_2} e^{i2\pi\eta_{C_1}^{im}}, \phi_{C_2} e^{i2\pi\phi_{C_2}^{im}}, \psi_{C_2} e^{i2\pi\psi_{C_2}^{im}}, r_{C_1} e^{i2\pi r_{C_2}^{im},}\right)$ be two CrC-PiFNs. Then

(1) if $SC\left(P_1\right) > SC\left(P_2\right)$, then $P_1 > P_2$,

(2) if $SC\left(P_1\right) = SC\left(P_2\right)$ then

(i) if $AC\left(P_1\right) > AC\left(P_2\right)$, then $P_1 > P_2$,

(ii) if $AC\left(P_1\right) = AC\left(P_2\right)$, then $P_1 = P_2$.

2.2 Algebraic circular complex picture fuzzy operational laws

The new direct sum, direct product, and scalar multiplication operations are defined for CrC-PiFSs in this subsection.

Let $P_1 = \left(\eta_{C_1} e^{i2\pi\eta_{C_1}^{im}}, \phi_{C_1} e^{i2\pi\phi_{C_1}^{im}}, \psi_{C_1} e^{i2\pi\psi_{C_1}^{im}}, r_{C_1} e^{i2\pi r_{C_1}^{im},}\right)$,

$P_2 = \left(\eta_{C_2} e^{i2\pi\eta_{C_1}^{im}}, \phi_{C_2} e^{i2\pi\phi_{C_2}^{im}}, \psi_{C_2} e^{i2\pi\psi_{C_2}^{im}}, r_{C_1} e^{i2\pi r_{C_2}^{im},}\right)$ be two CrC-PiFNs. Then we define the following algebraic circular complex picture fuzzy operational laws:

$$\text{(i) } P_1 \oplus^1 P_2 = \begin{pmatrix} \left(\frac{1}{2}\left((\eta_{C_1}) + (\eta_{C_2})\right)\right).e^{i2\pi\left(\frac{1}{2}\left(\left(\eta_{C_1}^{+im}\right)+\left(\eta_{C_2}^{+im}\right)\right)\right)}, \\ \left(1-\frac{1}{2}\left((\phi_{C_1}) + (\phi_{C_2})\right)\right).e^{i2\pi\left(1-\frac{1}{2}\left(\left(\phi_{C_1}^{+im}\right)+\left(\phi_{C_2}^{+im}\right)\right)\right)}, \\ \left(1-\frac{1}{2}\left((\psi_{C_1}) + (\psi_{C_2})\right)\right).e^{i2\pi\left(1-\frac{1}{2}\left(\left(\psi_{C_1}^{+im}\right)+\left(\psi_{C_2}^{+im}\right)\right)\right)}, \\ \left(\frac{1}{2}\left((r_{C_1}) + (r_{C_2})\right)\right).e^{i2\pi\left(\frac{1}{2}\left(\left(r_{C_1}^{+im}\right)+\left(r_{C_2}^{+im}\right)\right)\right)} \end{pmatrix};$$

$$\text{(ii) } P_1 \oplus^2 P_2 = \begin{pmatrix} \left(\frac{1}{2}\left((\eta_{C_1}) + (\eta_{C_2})\right)\right).e^{i2\pi\left(\frac{1}{2}\left(\left(\eta_{C_1}^{+im}\right)+\left(\eta_{C_2}^{+im}\right)\right)\right)}, \\ \left(1-\frac{1}{2}\left((\phi_{C_1}) + (\phi_{C_2})\right)\right).e^{i2\pi\left(1-\frac{1}{2}\left(\left(\phi_{C_1}^{+im}\right)+\left(\phi_{C_2}^{+im}\right)\right)\right)}, \\ \left(1-\frac{1}{2}\left((\psi_{C_1}) + (\psi_{C_2})\right)\right).e^{i2\pi\left(1-\frac{1}{2}\left(\left(\psi_{C_1}^{+im}\right)^q+\left(\psi_{C_2}^{+im}\right)^q\right)\right)}, \\ \left(1-\frac{1}{2}\left((r_{C_1}) + (r_{C_2})\right)\right).e^{i2\pi\left(1-\frac{1}{2}\left(\left(r_{C_1}^{+im}\right)+\left(r_{C_2}^{+im}\right)\right)\right)} \end{pmatrix}$$

$$\text{(iii) } P_1 \otimes^1 P_2 = \begin{pmatrix} \left(1-\frac{1}{2}\left((\eta_{C_1}) + (\eta_{C_2})\right)\right).e^{i2\pi\left(\frac{1}{2}\left(\left(\eta_{C_1}^{+im}\right)+\left(\eta_{C_2}^{+im}\right)\right)\right)}, \\ \left[\left(\frac{1}{2}\left((\phi_{C_1}) + (\phi_{C_2})\right)\right)\right].e^{i2\pi\left(\frac{1}{2}\left(\left(\phi_{C_1}^{+im}\right)+\left(\phi_{C_2}^{+im}\right)\right)\right)}, \\ \left[\left(\frac{1}{2}\left((\psi_{C_1}) + (\psi_{C_2})\right)\right)\right].e^{i2\pi\left(\frac{1}{2}\left(\left(\psi_{C_1}^{+im}\right)+\left(\psi_{C_2}^{+im}\right)\right)\right)}, \\ \left(1-\frac{1}{2}\left((r_{C_1}) + (r_{C_2})\right)\right) e^{i2\pi\left(\frac{1}{2}\left(\left(r_{C_1}^{+im}\right)+\left(r_{C_2}^{+im}\right)\right)\right)}, \end{pmatrix};$$

$$\text{(iii) } P_1 \otimes^2 P_2 = \begin{pmatrix} \left(1-\frac{1}{2}\left((\eta_{C_1}) + (\eta_{C_2})\right)\right).e^{i2\pi\left(\frac{1}{2}\left(\left(\eta_{C_1}^{+im}\right)+\left(\eta_{C_2}^{+im}\right)\right)\right)}, \\ \left[\left(\frac{1}{2}\left((\phi_{C_1}) + (\phi_{C_2})\right)\right)\right].e^{i2\pi\left(\frac{1}{2}\left(\left(\phi_{C_1}^{+im}\right)+\left(\phi_{C_2}^{+im}\right)\right)\right)}, \\ \left[\left(\frac{1}{2}\left((\psi_{C_1}) + (\psi_{C_2})\right)\right)\right].e^{i2\pi\left(\frac{1}{2}\left(\left(\psi_{C_1}^{+im}\right)+\left(\psi_{C_2}^{+im}\right)\right)\right)}, \\ \left[\left(\frac{1}{2}\left((r_{C_1}) + (r_{C_2})\right)\right)\right].e^{i2\pi\left(\frac{1}{2}\left(\left(r_{C_1}^{+im}\right)+\left(r_{C_2}^{+im}\right)\right)\right)} \end{pmatrix};$$

$$\text{(iii) } \alpha^1 P = \begin{pmatrix} (\alpha)\,\eta_C e^{i2\pi(\alpha)\eta_C^{+im}}, \left(\alpha\left(1-(\phi_C)\right)\right).e^{i2\pi\left(\alpha\left(1-\left(\phi_C^{+im}\right)\right)\right)}, \\ \left(\alpha\left(1-(\psi_C)\right)\right).e^{i2\pi\left(\alpha\left(1-\left(\psi_C^{+im}\right)\right)\right)}, (\alpha)\,r_C.e^{i2\pi(\alpha)r_C^{+im}} \end{pmatrix}, \text{ where } 0 \leq \alpha \leq 1;$$

$$\text{(iii) } \alpha^2 P = \begin{pmatrix} (\alpha)\,\eta_C e^{i2\pi(\alpha)\eta_C^{+im}}, \left(\alpha\left(1-(\phi_C)\right)\right).e^{i2\pi\left(\alpha\left(1-\left(\phi_C^{+im}\right)\right)\right)}, \\ \left(\alpha\left(1-(\psi_C)\right)\right).e^{i2\pi\left(\alpha\left(1-\left(\psi_C^{+im}\right)\right)\right)}, \\ \left(\alpha\left(1-(r_C)\right)\right).e^{i2\pi\left(\alpha\left(1-\left(r_C^{+im}\right)\right)\right)} \end{pmatrix}, \text{ where } 0 \leq \alpha \leq 1;$$

$$\text{(iv) } P^\lambda = \begin{pmatrix} l\,(\eta_C)^\lambda.e^{i2\pi r^{\frac{1}{q}}\left(\eta_C^{+im}\right)^\lambda}, l\,(\phi_C)^\lambda.e^{i2\pi r\left(\phi_C^{+im}\right)^\lambda} \\ l\,(\psi_C)^\lambda.e^{i2\pi r\left(\psi_C^{+im}\right)^\lambda}, l\,(r_C)^\lambda.e^{i2\pi r\left(r_C^{+im}\right)^\lambda} \end{pmatrix}$$

, where l is the total number of CrC-PiFNs that are a part of the procedure;

$$\text{(v) } P^{\odot 1\alpha} = \begin{pmatrix} \left(\alpha\left(1-(\eta_C)\right)\right).e^{i2\pi\left(\alpha\left(1-\left(\eta_C^{+im}\right)\right)\right)}, (\alpha)\,\phi_C.e^{i2\pi(\alpha)\psi_C^{+im}}, \\ (\alpha)\,\psi_C.e^{i2\pi(\alpha)\psi_C^{+im}}, \left(\alpha\left(1-(r_C)\right)\right).e^{i2\pi\left(\alpha\left(1-\left(r_C^{+im}\right)\right)\right)} \end{pmatrix},$$

where $0 \leq \alpha \leq 1$.

$$\text{(vi) } P^{\odot 2\alpha} = \begin{pmatrix} \left(\alpha\left(1-(\eta_C)\right)\right).e^{i2\pi\left(\alpha\left(1-\left(\eta_C^{+im}\right)\right)\right)}, (\alpha)\,\phi_C.e^{i2\pi(\alpha)\psi_C^{+im}}, \\ (\alpha)\,\psi_C.e^{i2\pi(\alpha)\psi_C^{+im}}, (\alpha)\,r_C.e^{i2\pi(\alpha)r_C^{+im}} \end{pmatrix},$$

where $0 \leq \alpha \leq 1$.

Let $\left\{P_j = \left(\eta_{C_j} e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_j} e^{i2\pi\phi_{C_r}^{+im}}, \psi_{C_j} e^{i2\pi\psi_{C_r}^{+im}}, r_{C_j} e^{i2\pi r_{C_j}^{+im},}\right) : j = 1, 2, ..., m\right\}$ be the collection of C_rC-PiF values and let C_rC-PiFWAM : $\Omega^m \to \Omega$. If

C_rC-PiFWAM$_E\left(P_1,P_2,P_3,...,P_m\right) = \left((\alpha_1^1P_1)^\lambda \oplus (\alpha_2^1P_2)^\lambda \oplus (\alpha_3^1P_3)^\lambda \oplus ... \oplus (\alpha_m^1P_m)^\lambda\right)^{\frac{1}{\lambda}}$ then C_rC-PiFWAM is called a circular complex picture fuzzy weighted averaging mean operator of dimension n, where Ω is the set of all C_rC-PiF values, $E=(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0,1]$, $\Sigma_{r=1}^m \alpha_r = 1$, where $r=1,2,...,m$.

Let $\left\{P_j = \left(\eta_{C_j}e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_j}e^{i2\pi\phi_{C_r}^{+im}}, \psi_{C_j}e^{i2\pi\psi_{C_r}^{+im}}, r_{C_j}e^{i2\pi r_{C_j}^{+im},}\right) : j=1,2,...,m\right\}$ be the collection of C_rC-PiF values. Then by using the C_rC-PiFWAM$_\text{E}$ operator their aggregated value is also a C_rC-PiSF value and

$$C_r^1C\text{-PiFWAM}_E\left(P_1,P_2,P_3,...,P_m\right)$$
$$= \begin{pmatrix} \left(\left(\sum_{r=1}^m\left(\alpha_j\left(\eta_{C_j}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(\sum\limits_{j=1}^m\left(\alpha_r\left(\eta_{C_j}^{+im}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1-\sum\limits_{j=1}^m\left(\alpha_j\left(1-\left(\phi_{C_j}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(1-\sum\limits_{j=1}^m\left(\alpha_j\left(1-\left(\phi_{C_j}^{+im}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1-\sum\limits_{j=1}^m\left(\alpha_j\left(1-\left(\psi_{C_j}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(1-\sum\limits_{r=1}^m\left(\alpha_j\left(1-\left(\psi_{C_j}^{+im}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(\sum\limits_{j=1}^m\left(\alpha_j\left(r_{C_j}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(\sum\limits_{j=1}^m\left(\alpha_j\left(r_{C_j}^{+im}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)} \end{pmatrix}.$$

$E=(\alpha_1,\ \alpha_2,\ ...,\ \alpha_n)^T$ are a weight vectors of P_r with $\alpha_r \in [0,1]$ and $\Sigma_{r=1}^n \alpha_r = 1, r=1,2,...,m.$

Let

$$\alpha_1^1P_1 = \begin{pmatrix} (\alpha_1)\eta_{C_1}.e^{i2\pi(\alpha_1)\eta_{C_1}^{+im}}, \left(\alpha_1\left(1-\left(\phi_{C_1}\right)\right)\right).e^{i2\pi\left(\alpha_1\left(1-\left(\phi_{C_1}^{+im}\right)\right)\right)}, \\ \left(\alpha_1\left(1-\left(\psi_{C_1}\right)\right)\right).e^{i2\pi\left(\alpha_1\left(1-\left(\psi_{C_1}^{+im}\right)\right)\right)}, (\alpha_1)r_{C_1}.e^{i2\pi(\alpha_1)r_{C_1}^{+im}} \end{pmatrix}$$

and

$$\alpha_2^1P_2 = \begin{pmatrix} (\alpha_2)\eta_{C_2}e^{i2\pi(\alpha_2)\eta_{C_2}^{+im}}, \left(\alpha_2\left(1-\left(\phi_{C_2}\right)\right)\right)e^{i2\pi\left(\alpha_2\left(1-\left(\phi_{C_2}^{+im}\right)\right)\right)}, \\ \left(\alpha_2\left(1-\left(\psi_{C_2}\right)\right)\right)e^{i2\pi\left(\alpha_2\left(1-\left(\psi_{C_2}^{+im}\right)\right)\right)}, (\alpha_2)r_{C_2}e^{i2\pi(\alpha_2)r_{C_2}^{+im}} \end{pmatrix}$$

Then

$$\left(\alpha_1^1P_1\right)^\lambda = \begin{pmatrix} l\left((\alpha_1)\eta_{C_1}\right)^\lambda e^{i2\pi l\left((\alpha_1)\eta_{C_1}^{+im}\right)^\lambda}, \\ l\left(\left(\alpha_1\left(1-\left(\phi_{C_1}\right)\right)\right)\right)^\lambda e^{i2\pi l\left(\left(\alpha_1\left(1-\left(\phi_{C_1}^{+im}\right)\right)\right)\right)^\lambda}, \\ l\left(\left(\alpha_1\left(1-\left(\psi_{C_1}\right)\right)\right)\right)^\lambda e^{i2\pi l\left(\left(\alpha_1\left(1-\left(\psi_{C_1}^{+im}\right)\right)\right)\right)^\lambda}, \\ l\left((\alpha_1)r_{C_1}\right)^\lambda e^{i2\pi l\left((\alpha_1)r_{C_1}^{+im}\right)^\lambda} \end{pmatrix}$$

and

$$\left(\alpha_2^1P_2\right)^\lambda = \begin{pmatrix} l\left((\alpha_2)\eta_{C_2}\right)^\lambda e^{i2\pi l\left((\alpha_2)\eta_{C_2}^{+im}\right)^\lambda}, \\ l\left(\left(\alpha_2\left(1-\left(\phi_{C_2}\right)\right)\right)\right)^\lambda e^{i2\pi l\left(\left(\alpha_2\left(1-\left(\phi_{C_2}^{+im}\right)\right)\right)\right)^\lambda}, \\ l\left(\left(\alpha_2\left(1-\left(\psi_{C_2}\right)\right)\right)\right)^\lambda e^{i2\pi l\left(\left(\alpha_2\left(1-\left(\psi_{C_2}^{+im}\right)\right)\right)\right)^\lambda}, \\ l\left((\alpha_2)r_{C_2}\right)^\lambda e^{i2\pi l\left((\alpha_2)r_{C_2}^{+im}\right)^\lambda} \end{pmatrix}.$$

Now

$$\left(\alpha_1^1 P_1\right)^{\lambda} \oplus \left(\alpha_2^1 P_2\right)^{\lambda} = \begin{pmatrix} \left(\frac{1}{2}\begin{pmatrix} \left(2\left(\left(\alpha_1\right)\eta_{C_1}\right)^{\lambda}\right) \\ +\left(2\left(\left(\alpha_2\right)\eta_{C_2}\right)^{\lambda}\right) \end{pmatrix}\right). e^{i2\pi\left(\frac{1}{2}\begin{pmatrix} \left(2\left(\left(\alpha_1\right)\eta_{C_1}^{+im}\right)^{\lambda}\right) \\ +\left(2\left(\left(\alpha_2\right)\eta_{C_2}^{+im}\right)^{\lambda}\right) \end{pmatrix}\right)}, \\ \left(1-\frac{1}{2}\begin{pmatrix} \left(2\left(\left(\alpha_1\left(1-\left(\phi_{C_1}\right)\right)\right)\right)^{\lambda}\right) \\ +\left(2\left(\left(\alpha_2\left(1-\left(\phi_{C_2}\right)\right)\right)\right)^{\lambda}\right) \end{pmatrix}\right). e^{i2\pi\left(1-\frac{1}{2}\begin{pmatrix} \left(2\left(\left(\alpha_1\left(1-\left(\phi_{C_1}^{+im}\right)\right)\right)\right)^{\lambda}\right) \\ +\left(2\left(\left(\alpha_2\left(1-\left(\phi_{C_2}^{+im}\right)\right)\right)\right)^{\lambda}\right) \end{pmatrix}\right)}, \\ \left(1-\frac{1}{2}\begin{pmatrix} \left(2\left(\left(\alpha_1\left(1-\left(\psi_{C_1}\right)\right)\right)\right)^{\lambda}\right) \\ +\left(2\left(\left(\alpha_2\left(1-\left(\psi_{C_2}\right)\right)\right)\right)^{\lambda}\right) \end{pmatrix}\right). e^{i2\pi\left(1-\frac{1}{2}\begin{pmatrix} \left(2\left(\left(\alpha_1\left(1-\left(\psi_{C_1}^{+im}\right)\right)\right)\right)^{\lambda}\right) \\ +\left(2\left(\left(\alpha_2\left(1-\left(\psi_{C_2}^{+im}\right)\right)\right)\right)^{\lambda}\right) \end{pmatrix}\right)}, \\ \left(\frac{1}{2}\begin{pmatrix} \left(2\left(\left(\alpha_1\right)\eta_{C_1}\right)^{\lambda}\right) \\ +\left(2\left(\left(\alpha_2\right)\eta_{C_2}\right)^{\lambda}\right) \end{pmatrix}\right). e^{i2\pi\left[\left(\frac{1}{2}\begin{pmatrix} \left(2\left(\left(\alpha_1\right)\eta_{C_1}^{+im}\right)^{\lambda}\right) \\ +\left(2\left(\left(\alpha_2\right)\eta_{C_2}^{+im}\right)^{\lambda}\right) \end{pmatrix}\right)\right]} \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{1}{2}\begin{pmatrix} 2\left(\left(\left(\alpha_1\right)\eta_{C_1}\right)^{\lambda}\right) \\ +2\left(\left(\left(\alpha_2\right)\eta_{C_2}\right)^{\lambda}\right) \end{pmatrix}\right).e^{i2\pi\left(\frac{1}{2}\begin{pmatrix} 2\left(\left(\left(\alpha_1\right)\eta_{C_1}^{+im}\right)^{\lambda}\right) \\ +2\left(\left(\left(\alpha_2\right)\eta_{C_2}^{+im}\right)^{\lambda}\right) \end{pmatrix}\right)}, \\ \left(1-\frac{1}{2}\begin{pmatrix} 2\left(\left(\left(\alpha_1\left(1-\left(\phi_{C_1}\right)\right)\right)\right)^{\lambda}\right) \\ +2\left(\left(\left(\alpha_2\left(1-\left(\phi_{C_2}\right)\right)\right)\right)^{\lambda}\right) \end{pmatrix}\right).e^{i2\pi\left(1-\frac{1}{2}\begin{pmatrix} 2\left(\left(\left(\alpha_1\left(1-\left(\phi_{C_1}^{+im}\right)\right)\right)\right)^{\lambda}\right) \\ +2\left(\left(\left(\alpha_2\left(1-\left(\phi_{C_2}^{+im}\right)\right)\right)\right)^{\lambda}\right) \end{pmatrix}\right)}, \\ \left(1-\frac{1}{2}\begin{pmatrix} 2\left(\left(\left(\alpha_1\left(1-\left(\psi_{C_1}\right)\right)\right)\right)^{\lambda}\right) \\ +2\left(\left(\left(\alpha_2\left(1-\left(\psi_{C_2}\right)\right)\right)\right)^{\lambda}\right) \end{pmatrix}\right).e^{i2\pi\left(1-\frac{1}{2}\begin{pmatrix} 2\left(\left(\left(\alpha_1\left(1-\left(\psi_{C_1}^{+im}\right)\right)\right)\right)^{\lambda}\right) \\ +2\left(\left(\left(\alpha_2\left(1-\left(\psi_{C_2}^{+im}\right)\right)\right)\right)^{\lambda}\right) \end{pmatrix}\right)} \\ \left(\frac{1}{2}\begin{pmatrix} 2\left(\left(\left(\alpha_1\right)r_{C_1}\right)^{\lambda}\right) \\ +2\left(\left(\left(\alpha_2\right)r_{C_2}\right)^{\lambda}\right) \end{pmatrix}\right).e^{i2\pi\left(\frac{1}{2}\begin{pmatrix} 2\left(\left(\left(\alpha_1\right)r_{C_1}^{+im}\right)^{\lambda}\right) \\ +2\left(\left(\left(\alpha_2\right)r_{C_2}^{+im}\right)^{\lambda}\right) \end{pmatrix}\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} \left(\left((\alpha_1)\,\eta_{C_1}^{+}\right)^{\lambda}\right) + \\ \left(\left((\alpha_2)\,\eta_{C_2}^{+}\right)^{\lambda}\right) \end{pmatrix} .e^{i2\pi\left[\begin{pmatrix} \left(\left((\alpha_1)\,\eta_{C_1}^{-im}\right)^{\lambda}\right) + \\ \left(\left((\alpha_2)\,\eta_{C_2}^{+im}\right)^{\lambda}\right) \end{pmatrix}\right]}, \\ \left(1 - \begin{pmatrix} \left(\alpha_1\left(1-\left(\phi_{C_1}(u)\right)\right)\right)^{\lambda} \\ +\left(\alpha_2\left(1-\left(\phi_{C_2}\right)\right)\right)^{\lambda} \end{pmatrix}\right) .e^{i2\pi\left(1-\begin{pmatrix} \left(\alpha_1\left(1-\left(\phi_{C_1}^{+im}\right)\right)\right)^{\lambda} \\ +\left(\alpha_2\left(1-\left(\phi_{C_2}^{+im}\right)\right)\right)^{\lambda} \end{pmatrix}\right)} \\ \left(1 - \begin{pmatrix} \left(\alpha_1\left(1-\left(\psi_{C_1}(u)\right)\right)\right)^{\lambda} \\ +\left(\alpha_2\left(1-\left(\psi_{C_2}\right)\right)\right)^{\lambda} \end{pmatrix}\right) .e^{i2\pi\left(1-\begin{pmatrix} \left(\alpha_1\left(1-\left(\psi_{C_1}^{+im}\right)\right)\right)^{\lambda} \\ +\left(\alpha_2\left(1-\left(\psi_{C_2}^{+im}\right)\right)\right)^{\lambda} \end{pmatrix}\right)}, \\ \begin{pmatrix} \left(\left((\alpha_1)\,r_{C_1}^{+}\right)^{\lambda}\right) + \\ \left(\left((\alpha_2)\,r_{C_2}^{+}\right)^{\lambda}\right) \end{pmatrix} .e^{i2\pi\left[\begin{pmatrix} \left(\left((\alpha_1)\,r_{C_1}^{+im}\right)^{\lambda}\right) + \\ \left(\left((\alpha_2)\,r_{C_2}^{+im}\right)^{\lambda}\right) \end{pmatrix}\right]} \end{pmatrix}$$

$$= \begin{pmatrix} \left(\sum_{j=1}^{2}\left(\alpha_j\left(\eta_{C_j}^{+}\right)\right)^{\lambda}\right) .e^{i2\pi\left(\sum_{j=1}^{2}\left(\alpha_j\left(\eta_{C_j}^{+im}\right)\right)^{\lambda}\right)}, \\ \left(1-\sum_{j=1}^{2}\left(\alpha_j\left(1-\left(\phi_{C_j}\right)\right)\right)^{\lambda}\right) .e^{i2\pi\left(1-\sum_{j=1}^{2}\left(\alpha_j\left(1-\left(\phi_{C_j}^{+im}\right)\right)\right)^{\lambda}\right)}, \\ \left(1-\sum_{j=1}^{2}\left(\alpha_j\left(1-\left(\psi_{C_j}\right)\right)\right)^{\lambda}\right) .e^{i2\pi\left(1-\sum_{j=1}^{2}\left(\alpha_j\left(1-\left(\psi_{C_j}^{+im}\right)\right)\right)^{\lambda}\right)}, \\ \left(\sum_{j=1}^{2}\left(\alpha_j\left(r_{C_j}\right)\right)^{\lambda}\right) .e^{i2\pi\left(\sum_{j=1}^{2}\left(\alpha_j\left(r_{C_j}^{+im}\right)\right)^{\lambda}\right)} \end{pmatrix}.$$

Let suppose that the result is true for $j = k$.

$$\left(\left(\alpha_1^1 P_1\right)^{\lambda} \oplus \left(\alpha_2^1 P_2\right)^{\lambda} \oplus \left(\alpha_3^1 P_3\right)^{\lambda} \oplus \ldots \oplus \left(\alpha_k^1 P_k\right)^{\lambda}\right)$$

$$= \begin{pmatrix} \left(\sum_{j=1}^{k}\left(\alpha_j\left(\eta_{C_j}\right)\right)^{\lambda}\right) .e^{i2\pi\left(\sum_{j=1}^{k}\left(\alpha_j\left(\eta_{C_j}^{+im}\right)\right)^{\lambda}\right)}, \\ \left(1-\sum_{j=1}^{k}\left(\alpha_j\left(1-\left(\phi_{C_j}\right)\right)\right)^{\lambda}\right) .e^{i2\pi\left(1-\sum_{j=1}^{k}\left(\alpha_j\left(1-\left(\phi_{C_j}^{+im}\right)\right)\right)^{\lambda}\right)}, \\ \left(1-\sum_{j=1}^{k}\left(\alpha_j\left(1-\left(\psi_{C_j}\right)\right)\right)^{\lambda}\right) .e^{i2\pi\left(1-\sum_{j=1}^{k}\left(\alpha_j\left(1-\left(\psi_{C_j}^{+im}\right)\right)\right)^{\lambda}\right)}, \\ \left(\sum_{j=1}^{k}\left(\alpha_j\left(r_{C_j}\right)\right)^{\lambda}\right) .e^{i2\pi\left(\sum_{j=1}^{k}\left(\alpha_j\left(r_{C_j}^{+im}\right)\right)^{\lambda}\right)} \end{pmatrix}.$$

We shows that the result is true for $j = k + 1$. So

$$\left((\alpha_1 P_1)^{\lambda} \oplus (\alpha_2 P_2)^{\lambda} \oplus (\alpha_3 P_3)^{\lambda} \oplus ... \oplus (\alpha_k P_k)^{\lambda}\right) \oplus (\alpha_{k+1} P_{k+1})^{\lambda}$$

$$= \begin{pmatrix} \left(\sum_{j=1}^{k} \left(\alpha_j \left(\eta_{C_j}^{+}\right)\right)^{\lambda}\right).e^{i2\pi\left(\sum_{j=1}^{k}\left(\alpha_j\left(\eta_{C_j}^{+im}\right)\right)^{\lambda}\right)}, \\ \left(1 - \sum_{j=1}^{k} \left(\alpha_j \left(1 - \left(\phi_{C_j}\right)\right)\right)^{\lambda}\right).e^{i2\pi\left(1-\sum_{j=1}^{k}\left(\alpha_j\left(1-\left(\phi_{C_j}^{+im}\right)\right)\right)^{\lambda}\right)}, \\ \left(1 - \sum_{j=1}^{k} \left(\alpha_j \left(1 - \left(\psi_{C_j}\right)\right)\right)^{\lambda}\right).e^{i2\pi\left(1-\sum_{j=1}^{k}\left(\alpha_j\left(1-\left(\psi_{C_j}^{+im}\right)\right)\right)^{\lambda}\right)}, \\ \left(\sum_{j=1}^{k} \left(\alpha_j \left(r_{C_j}\right)\right)^{\lambda}\right).e^{i2\pi\left(\sum_{j=1}^{k}\left(\alpha_j\left(r_{C_j}^{+im}\right)\right)^{\lambda}\right)} \end{pmatrix} \oplus$$

$$\left(\begin{pmatrix} l\left((\alpha_{k+1})\,\eta_{C_{k+1}}\right)^{\lambda} e^{i2\pi l\left((\alpha_{k+1})\eta_{C_{k+1}}^{+im}\right)^{\lambda}}, \\ l\left(\left(\alpha_{k+1}\left(1-\left(\phi_{C_{k+1}}\right)\right)\right)\right)^{\lambda} e^{i2\pi l\left(\left(\alpha_{k+1}\left(1-\left(\phi_{C_{k+1}}^{+im}\right)\right)\right)\right)^{\lambda}}, \\ l\left(\left(\alpha_{k+1}\left(1-\left(\psi_{C_{k+1}}\right)\right)\right)\right)^{\lambda} e^{i2\pi l\left(\left(\alpha_{1}\left(1-\left(\psi_{C_{k+1}}^{+im}\right)\right)\right)\right)^{\lambda}}, \\ l\left((\alpha_{k+1})\,r_{C_{k+1}}\right)^{\lambda} e^{i2\pi l\left((\alpha_{1})r_{C_{k+1}}^{+im}\right)^{\lambda}} \end{pmatrix}\right)$$

$$= \begin{pmatrix} \left(\sum_{j=1}^{k+1} \left(\alpha_j \left(\eta_{C_j}^{+}\right)\right)^{\lambda}\right).e^{i2\pi\left(\sum_{j=1}^{k+1}\left(\alpha_j\left(\eta_{C_j}^{+im}\right)\right)^{\lambda}\right)}, \\ \left(1 - \sum_{j=1}^{k+1} \left(\alpha_j \left(1 - \left(\phi_{C_j}\right)\right)\right)^{\lambda}\right).e^{i2\pi\left(1-\sum_{j=1}^{k+1}\left(\alpha_j\left(1-\left(\phi_{C_j}^{+im}\right)\right)\right)^{\lambda}\right)}, \\ \left(1 - \sum_{j=1}^{k+1} \left(\alpha_j \left(1 - \left(\psi_{C_j}\right)\right)\right)^{\lambda}\right).e^{i2\pi\left(1-\sum_{j=1}^{k+1}\left(\alpha_j\left(1-\left(\psi_{C_j}^{+im}\right)\right)\right)^{\lambda}\right)}, \\ \left(\sum_{j=1}^{k+1} \left(\alpha_j \left(r_{C_j}\right)\right)^{\lambda}\right).e^{i2\pi\left(\sum_{j=1}^{k+1}\left(\alpha_j\left(r_{C_j}^{+im}\right)\right)^{\lambda}\right)} \end{pmatrix}.$$

Thus

$$\left(\left((\alpha_1 P_1)^{\lambda} \oplus (\alpha_2 P_2)^{\lambda} \oplus (\alpha_3 P_3)^{\lambda} \oplus ... \oplus (\alpha_k P_k)^{\lambda}\right) \oplus (\alpha_{k+1} P_{k+1})^{\lambda}\right)^{\frac{1}{\lambda}}$$

$$= \begin{pmatrix} \left(\left(\sum_{j=1}^{k+1}\left(\alpha_j\left(\eta_{C_j}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(\sum_{r=1}^{k+1}\left(\alpha_j\left(\eta_{C_j}^{+im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1-\sum_{j=1}^{k+1}\left(\alpha_j\left(1-\left(\phi_{C_j}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(1-\sum_{j=1}^{k+1}\left(\alpha_j\left(1-\left(\phi_{C_j}^{+im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1-\sum_{j=1}^{k+1}\left(\alpha_j\left(1-\left(\psi_{C_j}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(1-\sum_{j=1}^{k+1}\left(\alpha_j\left(1-\left(\psi_{C_j}^{+im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(\sum_{j=1}^{k+1}\left(\alpha_j\left(r_{C_j}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(\sum_{r=1}^{k+1}\left(\alpha_j\left(r_{C_j}^{+im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)} \end{pmatrix}.$$

Let $\left\{P_j = \left(\eta_{C_j} e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_j} e^{i2\pi\phi_{C_r}^{+im}}, \psi_{C_j} e^{i2\pi\psi_{C_r}^{+im}}, r_{C_j} e^{i2\pi r_{C_j}^{+im},}\right) : j = 1, 2, ..., m\right\}$ be the collection of $C_r C$-PiF values values. Then the $C_r^2 C\text{-}PiFWAM_E$ operator is defined by

$$C_r^2 CT - SPFWAM_E\left(P_1, P_2, P_3, ..., P_m\right)$$

$$= \begin{pmatrix} \left(\left(\sum_{r=1}^{m}\left(\alpha_j\left(\eta_{C_j}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(\sum_{j=1}^{m}\left(\alpha_r\left(\eta_{C_j}^{+im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1-\sum_{j=1}^{m}\left(\alpha_j\left(1-\left(\phi_{C_j}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(1-\sum_{j=1}^{m}\left(\alpha_j\left(1-\left(\phi_{C_j}^{+im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1-\sum_{j=1}^{m}\left(\alpha_j\left(1-\left(\psi_{C_j}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(1-\sum_{r=1}^{m}\left(\alpha_j\left(1-\left(\psi_{C_j}^{+im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1-\sum_{j=1}^{m}\left(\alpha_j\left(1-\left(r_{C_j}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(1-\sum_{r=1}^{m}\left(\alpha_j\left(1-\left(r_{C_j}^{+im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)} \end{pmatrix}.$$

$E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_n)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\Sigma_{r=1}^{n}\alpha_r = 1, r = 1, 2, ..., m.$

Similar to the proof of Theorem 2.2.

Let $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r} e^{i2\pi\phi_{C_r}^{+im}}, \psi_{C_r} e^{i2\pi\psi_{C_r}^{+im}}, r_{C_r} e^{i2\pi r_{C_r}^{+im},}\right) : r = 1, 2, ..., m\right\}$ be the collection of $C_r C\text{-}PiSF$ values. Let $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\Sigma_{r=1}^{m}\alpha_r = 1$. If $\left(\eta_{C_1} e^{i2\pi\eta_{C_1}^{+im}}, \phi_{C_1} e^{i2\pi\phi_{C_1}^{+im}}, \psi_{C_1} e^{i2\pi\psi_{C_1}^{+im}}, r_{C_1} e^{i2\pi r_{C_1}^{+im},}\right) =$
$\left(\eta_{C_2} e^{i2\pi\eta_{C_2}^{+im}}, \phi_{C2} e^{i2\pi\phi_{C_2}^{+im}}, \psi_{C_2} e^{i2\pi\psi_{C_2}^{+im}}, r_{C_2} e^{i2\pi r_{C_2}^{+im},}\right) = ...$
$= \left(\eta_{C_m} e^{i2\pi\eta_{C_m}^{+im}}, \phi_{C_m} e^{i2\pi\phi_{C_m}^{+im}}, \psi_{C_m} e^{i2\pi\psi_{C_m}^{+im}}, r_{C_m} e^{i2\pi r_{C_m}^{+im},}\right) =$
$\left(\eta_C e^{i2\pi\eta_C^{+im}}, \phi_C e^{i2\pi\phi_C^{+im}}, \psi_C e^{i2\pi\psi_C^{+im}}, r_C e^{i2\pi r_C^{+im},}\right)$ and $\lambda = 1,$
then $C_r^1 C\text{-}PiFWAM_E\left(P_1, P_2, P_3, ..., P_m\right) =$

$\left(\eta_C e^{i2\pi\eta_C^{+im}}, \phi_C e^{i2\pi\phi_C^{+im}}, \psi_C e^{i2\pi\psi_C^{+im}}, r_C e^{i2\pi r_C^{+im}},\right)$.

Let $\left(\eta_{C_1} e^{i2\pi\eta_{C_1}^{+im}}, \phi_{C_1} e^{i2\pi\phi_{C_1}^{+im}}, \psi_{C_1} e^{i2\pi\psi_{C_1}^{+im}}, r_{C_1} e^{i2\pi r_{C_1}^{+im}},\right)$,
$= \left(\eta_{C_2} e^{i2\pi\eta_{C_2}^{+im}}, \phi c_2 e^{i2\pi\phi_{C_2}^{+im}}, \psi_{C_2} e^{i2\pi\psi_{C_2}^{+im}}, r_{C_2} e^{i2\pi r_{C_2}^{+im}},\right) = ...$
$= \left(\eta_{C_m} e^{i2\pi\eta_{C_m}^{+im}}, \phi_{C_m} e^{i2\pi\phi_{C_m}^{+im}}, \psi_{C_m} e^{i2\pi\psi_{C_m}^{+im}}, r_{C_m} e^{i2\pi r_{C_m}^{+im}},\right)$
$= \left(\eta_C e^{i2\pi\eta_C^{+im}}, \phi_C e^{i2\pi\phi_C^{+im}}, \psi_C e^{i2\pi\psi_C^{+im}}, r_C e^{i2\pi r_C^{+im}},\right)$ and $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0,1]$ and $\Sigma_{r=1}^m \alpha_r = 1$, where $r = 1, 2, ..., m$. Based on Definition 2.2, we get

$$C_rC - PiFWAM_E\left(P_1, P_2, P_3, ..., P_m\right)$$

$$= \begin{pmatrix} \left(\left(\sum_{r=1}^m \left(\alpha_r\left(\eta_{C_r}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right) e^{i2\pi\left(\left(\sum_{r=1}^m \left(\alpha_r\left(\eta_{C_r}^{+im}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1-\sum_{r=1}^m \left(\alpha_r\left(1-\left(\phi_{C_r}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right) e^{i2\pi\left(\left(1-\sum_{r=1}^m \left(\alpha_r\left(1-\left(\phi_{C_r}^{+im}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1-\sum_{r=1}^m \left(\alpha_r\left(1-\left(\psi_{C_r}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right) e^{i2\pi\left(\left(1-\sum_{r=1}^m \left(\alpha_r\left(1-\left(\psi_{C_r}^{+im}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(\sum_{r=1}^m \left(\alpha_r\left(r_{C_r}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right) .e^{i2\pi\left(\left(\sum_{r=1}^m \left(\alpha_r\left(r_{C_r}^{+im}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \left(\sum_{r=1}^m \left(\alpha_r\left(\eta_C\right)\right)\right) .e^{i2\pi\left(\sum_{r=1}^m \left(\alpha_r\left(\eta_C^{+im}\right)\right)\right)}, \\ \left(1-\sum_{r=1}^m \left(\alpha_r\left(1-\left(\phi_C\right)\right)\right)\right) e^{i2\pi\left(1-\sum_{r=1}^m \left(\alpha_r\left(1-\left(\phi_C^{+im}\right)\right)\right)\right)}, \\ \left(1-\sum_{r=1}^m \left(\alpha_r\left(1-\left(\psi_C\right)\right)\right)\right) e^{i2\pi\left(1-\sum_{r=1}^m \left(\alpha_r\left(1-\left(\psi_C^{+im}\right)\right)\right)\right)}, \\ \left(\sum_{r=1}^m \left(\alpha_r\left(r_C\right)\right)\right) .e^{i2\pi\left(\sum_{r=1}^m \left(\alpha_r\left(r_C^{+im}\right)\right)\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \left(\left(\eta_C\right)\sum_{r=1}^m \left(\alpha_r\right)\right) .e^{i2\pi\left(\left(\eta_C^{+im}\right)\sum_{r=1}^m \left(\alpha_r\right)\right)}, \\ \left(1-\left(1-\left(\phi_C\right)\right)\sum_{r=1}^m \left(\alpha_r\right)\right) .e^{i2\pi\left(1-\left(1-\left(\phi_C^{+im}\right)\right)\sum_{r=1}^m \left(\alpha_r\right)\right)}, \\ \left(1-\left(1-\left(\psi_C\right)\right)\sum_{r=1}^m \left(\alpha_r\right)\right) .e^{i2\pi\left(1-\left(1-\left(\psi_C^{+im}\right)\right)\sum_{r=1}^m \left(\alpha_r\right)\right)}, \\ \left(\left(r_C\right)\sum_{r=1}^m \left(\alpha_r\right)\right) .e^{i2\pi\left(\left(r_C^{+im}\right)\sum_{r=1}^m \left(\alpha_r\right)\right)} \end{pmatrix}$$

$$= \begin{pmatrix} ((\eta_C))\, e^{i2\pi\left(\left(\eta_C^{+im}\right)\right)}, \\ (1-(1-(\phi_C)))\,.e^{i2\pi\left(1-\left(1-\left(\phi_C^{+im}\right)\right)\right)}, \\ (1-(1-(\psi_C)))\,.e^{i2\pi\left(1-\left(1-\left(\psi_C^{+im}\right)\right)\right)}, \\ ((r_C))\, e^{i2\pi\left(\left(r_C^{+im}\right)\right)} \end{pmatrix}$$

$$= \left(\eta_C.e^{i2\pi\eta_C^{+im}}, \phi_C.e^{i2\pi\phi_C^{+im}}, \psi_C.e^{i2\pi\psi_C^{+im}}, r_C.e^{i2\pi r_C^{+im}}\right).$$

Let $\left\{P_r = \left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}}, \psi_{C_r}e^{i2\pi\psi_{C_r}^{+im}}, r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ be the collection of C_rC-PiF values. Let $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0,1]$ and $\Sigma_{r=1}^m \alpha_r = 1$. If $\left(\eta_{C_1}e^{i2\pi\eta_{C_1}^{+im}}, \phi_{C_1}e^{i2\pi\phi_{C_1}^{+im}}, \psi_{C_1}e^{i2\pi\psi_{C_1}^{+im}}, r_{C_1}e^{i2\pi r_{C_1}^{+im},}\right) =$
$\left(\eta_{C_2}e^{i2\pi\eta_{C_2}^{+im}}, \phi c_2 e^{i2\pi\phi_{C_2}^{+im}}, \psi_{C_2}e^{i2\pi\psi_{C_2}^{+im}}, r_{C_2}e^{i2\pi r_{C_2}^{+im},}\right) = ...$
$= \left(\eta_{C_m}e^{i2\pi\eta_{C_m}^{+im}}, \phi_{C_m}e^{i2\pi\phi_{C_m}^{+im}}, \psi_{C_m}e^{i2\pi\psi_{C_m}^{+im}}, r_{C_m}e^{i2\pi r_{C_m}^{+im},}\right) =$
$\left(\eta_C e^{i2\pi\eta_C^{+im}}, \phi_C e^{i2\pi\phi_C^{+im}}, \psi_C e^{i2\pi\psi_C^{+im}}, r_C e^{i2\pi r_C^{+im},}\right)$ and $\lambda = 1$,
then C_r^2C-PiSFWAM$_E\,(P_1, P_2, P_3, ..., P_m) =$
$\left(\eta_C e^{i2\pi\eta_C^{+im}}, \phi_C e^{i2\pi\phi_C^{+im}}, \psi_C e^{i2\pi\psi_C^{+im}}, r_C e^{i2\pi r_C^{+im},}\right)$.

Similar to the proof of Theorem 2.2

Let $\left\{P_r = \left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}}, \psi_{C_r}e^{i2\pi\psi_{C_r}^{+im}}, r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ be the collection of C_rC-PiF values and $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of
P_r with $\alpha_r \in [0,1]$ and $\Sigma_{r=1}^m \alpha_r = 1$, where $r = 1, 2, ..., m$.

Let $P^- = \left(\min\limits_{1\le r\le m}\eta_{C_r}.e^{i2\pi\min\limits_{1\le r\le m}\eta_{C_r}^{+im}}, \max\limits_{1\le r\le m}\phi_{C_r}.e^{i2\pi\max\limits_{1\le r\le m}\phi_{C_r}^{+im}}, \max\limits_{1\le r\le m}\psi_{C_r}.e^{i2\pi\max\limits_{1\le r\le m}\psi_{C_r}^{+im}}, \min\limits_{1\le r\le m}r_{C_r}.e^{i2\pi\min\limits_{1\le r\le m}r_{C_r}^{+im}}\right)$

$P^+ = \left(\max\limits_{1\le r\le m}\eta_{C_r}.e^{i2\pi\max\limits_{1\le r\le m}\eta_{C_r}^{+im}}, \min\limits_{1\le r\le m}\phi_{C_r}.e^{i2\pi\max\limits_{1\le r\le m}\phi_{C_r}^{+im}}, \min\limits_{1\le r\le m}\psi_{C_r}.e^{i2\pi\min\limits_{1\le r\le m}\psi_{C_r}^{+im}}, \max\limits_{1\le r\le m}r_{C_r}.e^{i2\pi\max\limits_{1\le r\le m}r_{C_r}^{+im}},\right)$,

where and $\lambda = 1$. Then $P^- \le C_r^1C$-PiFWAM$_E\,(P_1, P_2, P_3, ..., P_m) \le P^+$.

Let $\left\{P_r = \left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}}, \psi_{C_r}e^{i2\pi\psi_{C_r}^{+im}}, r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ be the collection of C_rC-PiSF values and $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r, where $r = 1, 2, ..., m$ with $\alpha_r \in [0,1]$ and $\Sigma_{r=1}^m \alpha_r = 1$.

Let $P^- = \left(\min\limits_{1\le r\le m}\eta_{C_r}.e^{i2\pi\min\limits_{1\le r\le m}\eta_{C_r}^{+im}}, \max\limits_{1\le r\le m}\phi_{C_r}.e^{i2\pi\max\limits_{1\le r\le m}\phi_{C_r}^{+im}}, \max\limits_{1\le r\le m}\psi_{C_r}.e^{i2\pi\max\limits_{1\le r\le m}\psi_{C_r}^{+im}}, \min\limits_{1\le r\le m}r_{C_r}.e^{i2\pi\min\limits_{1\le r\le m}r_{C_r}^{+im}}\right)$

$P^+ = \left(\max\limits_{1\le r\le m}\eta_{C_r}.e^{i2\pi\max\limits_{1\le r\le m}\eta_{C_r}^{+im}}, \min\limits_{1\le r\le m}\phi_{C_r}.e^{i2\pi\max\limits_{1\le r\le m}\phi_{C_r}^{+im}}, \min\limits_{1\le r\le m}\psi_{C_r}.e^{i2\pi\min\limits_{1\le r\le m}\psi_{C_r}^{+im}}, \max\limits_{1\le r\le m}r_{C_r}.e^{i2\pi\max\limits_{1\le r\le m}r_{C_r}^{+im}},\right)$

where and $\lambda = 1$, we get $(\eta_{C_t})^q \le \left(\max\limits_{1\le r\le m}\eta_{C_r}\right)$ where $t = 1, 2, ..., m$. Then we have $\alpha_r\,(\eta_{C_t}) \le \alpha_r\left(\max\limits_{1\le r\le m}\eta_{C_r}\right)$. This implies that $\sum\limits_{r=1}^m \alpha_r\,(\eta_{C_t}) \le \sum\limits_{r=1}^m \alpha_r\left(\max\limits_{1\le r\le m}\eta_{C_r}\right)$ then we have, $\sum\limits_{r=1}^m \alpha_r\,(\eta_{C_t}) \le \left(\max\limits_{1\le r\le m}\eta_{C_r}\right)\sum\limits_{r=1}^m \alpha_r$ This implies that $\sum\limits_{r=1}^m \alpha_r\,(\eta_{C_t}) \le \left(\max\limits_{1\le r\le m}\eta_{C_r}\right)$ and this implies that $\left(\sum\limits_{r=1}^m \alpha_r\,(\eta_{C_t})\right) \le \max\limits_{1\le r\le m}\eta_{C_r}$. Hence
$\left(\sum\limits_{r=1}^m \alpha_r\,(\eta_{C_t})\right) \le \max\limits_{1\le r\le m}\eta_{C_r}$. Also
$\left(\sum\limits_{r=1}^m \alpha_r\,(\eta_{C_t}^{im})\right) \le \max\limits_{1\le r\le m}\eta_{C_r}^{im}$. This implies that

$e^{i2\pi\left(\sum_{r=1}^{m}\alpha_r\left(\eta_{C_t}^{im}\right)\right)} \leq e^{i2\pi \max_{1\leq r\leq m}\eta_{C_r}^{im}}$. Thus

$$\begin{aligned}&\left(\sum_{r=1}^{m}\alpha_r\left(\eta_{C_t}^{im}\right)\right).e^{i2\pi\left(\sum_{r=1}^{m}\alpha_r\left(\eta_{C_t}^{im}\right)\right)}\\ \leq\ & \max_{1\leq r\leq m}\eta_{C_r}^{im}.e^{i2\pi\max_{1\leq r\leq m}\eta_{C_r}^{im}}.\end{aligned}$$

On the other hand, we get $(\phi_{C_r}) \geq \left(\min_{1\leq r\leq m}\phi_C\right)$. This implies that $-(\phi_{C_r}) \leq -\left(\min_{1\leq r\leq m}\phi_C\right)$ and , also $1-(\phi_{C_r}) \leq 1-\left(\min_{1\leq r\leq m}\phi_C\right)$, where, $r=1,2,...,m$. Then we have $\alpha_r\left(1-\left(\phi_{C_r}(u)\right)\right) \leq \alpha_r\left(1-\left(\phi_C(u)\right)\right)$ $\alpha_r\left(1-\left(\phi_{C_r}\right)\right) \leq \alpha_r\left(1-\left(\min_{1\leq r\leq m}\phi_C\right)\right)$ then we get,

$$\begin{aligned}\alpha_r\left(1-\left(\phi_{C_r}\right)\right) &\leq \alpha_r\left(1-\left(\min_{1\leq r\leq m}\phi_C\right)\right)\\ &\Rightarrow \sum_{r=1}^{m}\alpha_r\left(1-\left(\phi_{C_r}\right)\right) \leq \sum_{r=1}^{m}\alpha_r\left(1-\left(\min_{1\leq r\leq m}\phi_C\right)\right)\\ &\Rightarrow \sum_{r=1}^{m}\alpha_r\left(1-\left(\phi_{C_r}\right)\right) \leq \left(1-\left(\min_{1\leq r\leq m}\phi_C\right)\right)\sum_{r=1}^{m}\alpha_r\\ &\Rightarrow -\sum_{r=1}^{m}\alpha_r\left(1-\left(\phi_{C_r}\right)\right) \geq -\left(1-\left(\min_{1\leq r\leq m}\phi_C\right)\right)\\ &\Rightarrow 1-\sum_{r=1}^{m}\alpha_r\left(1-\left(\phi_{C_r}\right)\right) \geq 1-\left(1-\left(\min_{1\leq r\leq m}\phi_C\right)\right)\\ &\Rightarrow 1-\sum_{r=1}^{m}\alpha_r\left(1-\left(\phi_{C_r}\right)\right) \geq \left(\min_{1\leq r\leq m}\phi_C\right)\\ &\Rightarrow \left(1-\sum_{r=1}^{m}\alpha_r\left(1-\left(\phi_{C_r}\right)\right)\right) \geq \min_{1\leq r\leq m}\phi_C.\end{aligned}$$

Also $\left(1-\sum_{r=1}^{m}\alpha_r\left(1-\left(\phi_{C_r}\right)\right)\right) \geq \min_{1\leq r\leq m}\phi_C$. This implies that
$\left(1-\sum_{r=1}^{m}\alpha_r\left(1-\left(\phi_{C_r}\right)\right)\right) \geq \min_{1\leq r\leq m}\phi_C$. Also
$\left(1-\sum_{r=1}^{m}\alpha_r\left(1-\left(\phi_{C_r}^{im}\right)\right)\right) \geq \min_{1\leq r\leq m}\phi_C^{+im}$. This implies that
$e^{i2\pi\left(1-\sum_{r=1}^{m}\alpha_r\left(1-\left(\phi_{C_r}^{+im}\right)\right)\right)} \geq e^{i2\pi\min_{1\leq r\leq m}\phi_C^{+im}}$. Thus

$$\begin{aligned}&\left(1-\sum_{r=1}^{m}\alpha_r\left(1-\left(\phi_{C_r}\right)\right)\right).e^{i2\pi\left(1-\sum_{r=1}^{m}\alpha_r\left(1-\left(\phi_{C_r}^{+im}\right)\right)\right)}\\ \geq\ & \min_{1\leq r\leq m}\phi_C.e^{i2\pi\min_{1\leq r\leq m}\phi_C^{-im}}.\end{aligned}$$

On the other hand, we get $(\psi_{C_r}) \geq \left(\min_{1\leq r\leq m}\psi_C\right)$. This implies that $-(\psi_{C_r}) \leq -\left(\min_{1\leq r\leq m}\psi_C\right)$ and , also $1-(\psi_{C_r}) \leq 1-\left(\min_{1\leq r\leq m}\psi_C\right)$, where, $r=1,2,...,m$. Then we have $\alpha_r\left(1-\left(\psi_{C_r}(u)\right)\right) \leq \alpha_r\left(1-\left(\psi_C(u)\right)\right)$

$\alpha_r\left(1-\left(\psi_{C_r}\right)\right) \leq \alpha_r\left(1-\left(\min\limits_{1\leq r\leq m}\psi_C\right)\right)$ then we get,

$$\begin{aligned}
\alpha_r\left(1-\left(\psi_{C_r}\right)\right) &\leq \alpha_r\left(1-\left(\min_{1\leq r\leq m}\psi_C\right)\right)\\
&\Rightarrow \sum_{r=1}^{m}\alpha_r\left(1-\left(\psi_{C_r}\right)\right) \leq \sum_{r=1}^{m}\alpha_r\left(1-\left(\min_{1\leq r\leq m}\psi_C\right)\right)\\
&\Rightarrow \sum_{r=1}^{m}\alpha_r\left(1-\left(\psi_{C_r}\right)\right) \leq \left(1-\left(\min_{1\leq r\leq m}\psi_C\right)\right)\sum_{r=1}^{m}\alpha_r\\
&\Rightarrow -\sum_{r=1}^{m}\alpha_r\left(1-\left(\psi_{C_r}\right)\right) \geq -\left(1-\left(\min_{1\leq r\leq m}\psi_C\right)\right)\\
&\Rightarrow 1-\sum_{r=1}^{m}\alpha_r\left(1-\left(\psi_{C_r}\right)\right) \geq 1-\left(1-\left(\min_{1\leq r\leq m}\psi_C\right)\right)\\
&\Rightarrow 1-\sum_{r=1}^{m}\alpha_r\left(1-\left(\psi_{C_r}\right)\right) \geq \left(\min_{1\leq r\leq m}\psi_C\right)\\
&\Rightarrow \left(1-\sum_{r=1}^{m}\alpha_r\left(1-\left(\psi_{C_r}\right)\right)\right) \geq \min_{1\leq r\leq m}\psi_C.
\end{aligned}$$

Also $\left(1-\sum\limits_{r=1}^{m}\alpha_r\left(1-\left(\psi_{C_r}\right)\right)\right) \geq \min\limits_{1\leq r\leq m}\psi_C$. This implies that
$\left(1-\sum\limits_{r=1}^{m}\alpha_r\left(1-\left(\psi_{C_r}\right)\right)\right) \geq \min\limits_{1\leq r\leq m}\psi_C$. Also
$\left(1-\sum\limits_{r=1}^{m}\alpha_r\left(1-\left(\psi^{im}_{C_r}\right)\right)\right) \geq \min\limits_{1\leq r\leq m}\psi^{im}_C$. This implies that
$e^{i2\pi\left(1-\sum\limits_{r=1}^{m}\alpha_r\left(1-\left(\psi^{im}_{C_r}\right)\right)\right)} \geq e^{i2\pi\min\limits_{1\leq r\leq m}\psi^{im}_C}$. Thus

$$\begin{aligned}
&\left(1-\sum_{r=1}^{m}\alpha_r\left(1-\left(\psi_{C_r}\right)\right)\right).e^{i2\pi\left(1-\sum\limits_{r=1}^{m}\alpha_r\left(1-\left(\psi^{+im}_{C_r}\right)\right)\right)}\\
\geq\ &\min_{1\leq r\leq m}\psi_C.e^{i2\pi\min\limits_{1\leq r\leq m}\psi^{+im}_C}.
\end{aligned}$$

$\left(r_{C_t}\right) \leq \left(\max\limits_{1\leq r\leq m}r_{C_r}\right)$ where $t=1,2,\ldots,m$. Then we have $r_r\left(\eta_{C_t}\right) \leq r_r\left(\max\limits_{1\leq r\leq m}\eta_{C_r}\right)$. This implies that $\sum\limits_{r=1}^{m}\alpha_r\left(r_{C_t}\right) \leq \sum\limits_{r=1}^{m}\alpha_r\left(\max\limits_{1\leq r\leq m}r_{C_r}\right)$ then we have, $\sum\limits_{r=1}^{m}\alpha_r\left(r_{C_t}\right) \leq \left(\max\limits_{1\leq r\leq m}r_{C_r}\right)\sum\limits_{r=1}^{m}\alpha_r$ This implies that $\sum\limits_{r=1}^{m}\alpha_r\left(r_{C_t}\right) \leq \left(\max\limits_{1\leq r\leq m}r_{C_r}\right)$ and this implies that $\left(\sum\limits_{r=1}^{m}\alpha_r\left(r_{C_t}\right)\right) \leq \max\limits_{1\leq r\leq m}r_{C_r}$. Hence
$\left(\sum\limits_{r=1}^{m}\alpha_r\left(r_{C_t}\right)\right) \leq \max\limits_{1\leq r\leq m}r_{C_r}$. Also
$\left(\sum\limits_{r=1}^{m}\alpha_r\left(r^{im}_{C_t}\right)\right) \leq \max\limits_{1\leq r\leq m}r^{im}_{C_r}$. This implies that
$e^{i2\pi\left(\sum\limits_{r=1}^{m}\alpha_r\left(r^{im}_{C_t}\right)\right)} \leq e^{i2\pi\max\limits_{1\leq r\leq m}r^{im}_{C_r}}$. Thus

$$\begin{aligned}
&\left(\sum_{r=1}^{m}\alpha_r\left(r^{im}_{C_t}\right)\right).e^{i2\pi\left(\sum\limits_{r=1}^{m}\alpha_r\left(r^{im}_{C_t}\right)\right)}\\
\leq\ &\max_{1\leq r\leq m}r^{im}_{C_r}.e^{i2\pi\max\limits_{1\leq r\leq m}r^{im}_{C_r}}.
\end{aligned}$$

Based on Definition 2.2 and Definition 2.2, we get $CrC\text{-}PiFWAM_E\left(P_1,P_2,P_3,...,P_m\right) \leq P^+$. Similarly, we can have $C_r^1C\text{-}PiFWAM_E\left(P_1,P_2,P_3,...,P_m\right) \geq P^-$. Finally we get,

$$P^+ \geq C_r^1C\text{-}PiFWAM_E\left(P_1,P_2,P_3,...,P_m\right) \geq P^-.$$

Let $\left\{P_r = \left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}}, \psi_{C_r}e^{i2\pi\psi_{C_r}^{+im}}, r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right) : r = 1,2,...,m\right\}$ be the collection of $C_r^2C\text{-}PiF$ values and $E = \left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m\right)^T$ are a weight vectors of P_r with $\alpha_r \in [0,1]$ and $\Sigma_{r=1}^m \alpha_r = 1$, where $r = 1,2,...,m$.

Let

$$P^- = \left(\min_{1\leq r\leq m}\eta_{C_r}.e^{i2\pi\min\limits_{1\leq r\leq m}\eta_{C_r}^{+im}}, \max_{1\leq r\leq m}\phi_{C_r}.e^{i2\pi\max\limits_{1\leq r\leq m}\phi_{C_r}^{+im}}, \max_{1\leq r\leq m}\psi_{C_r}.e^{i2\pi\max\limits_{1\leq r\leq m}\psi_{C_r}^{+im}}, \max_{1\leq r\leq m}r_{C_r}.e^{i2\pi\min\limits_{1\leq r\leq m}r_{C_r}^{+im}}\right)$$

$$P^+ = \left(\max_{1\leq r\leq m}\eta_{C_r}.e^{i2\pi\max\limits_{1\leq r\leq m}\eta_{C_r}^{+im}}, \min_{1\leq r\leq m}\phi_{C_r}.e^{i2\pi\max\limits_{1\leq r\leq m}\phi_{C_r}^{+im}}, \min_{1\leq r\leq m}\psi_{C_r}.e^{i2\pi\min\limits_{1\leq r\leq m}\psi_{C_r}^{+im}}, \min_{1\leq r\leq m}r_{C_r}.e^{i2\pi\max\limits_{1\leq r\leq m}r_{C_r}^{+im}},\right),$$

where and $\lambda = 1$. Then $P^- \leq C_r^2C\text{-}PiFWAM_E\left(P_1,P_2,P_3,...,P_m\right) \leq P^+$.

Similar to the proof of Theorem 2.2.

Let $\left\{P_r = \left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}}, \psi_{C_r}e^{i2\pi\psi_{C_r}^{+im}}, r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right) : r = 1,2,...,m\right\}$ and $\left\{P_r^* = \left(\eta_{C_r}^*e^{i2\pi\eta_{C_r}^{+im*}}, \phi_{C_r}^*e^{i2\pi\phi_{C_r}^{+im*}}, \psi_{C_r}^*e^{i2\pi\psi_{C_r}^{+im*}}, r_{C_r}^*e^{i2\pi r_{C_r}^{+im*}}\right) : r = 1,2,...,m\right\}$ are two collections of $C_rC\text{-}PiF$ values. If $\eta_{C_r} \leq \eta_{C_r}^*$, $\eta_{C_r}^{im} \leq \eta_{C_r}^{im*}$, $\phi_{C_r} \geq \phi_{C_r}^*$, $\phi_{C_r}^{im} \geq \phi_{C_r}^{im*}$, $\psi_{C_r} \geq \psi_{C_r}^*$, $\psi_{C_r}^{im} \geq \psi_{C_r}^{im*}$, $r_{C_r} \leq r_{C_r}^*$, and $r_{C_r}^{im} \leq r_{C_r}^{im*}$ where $r = 1,2,...,m$, then $C_r^1C\text{-}PiFWAM_E\left(P_1,P_2,P_3,...,P_m\right) \leq C_r^1C\text{-}PiFWAM_E\left(P_1^*,P_2^*,P_3^*,...,P_m^*\right)$.

Given that $P_r = \left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}}, \psi_{C_r}e^{i2\pi\psi_{C_r}^{+im}}, r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right)$ and $P_r^* = \left(\eta_{C_r}^*e^{i2\pi\eta_{C_r}^{+im*}}, \phi_{C_r}^*e^{i2\pi\phi_{C_r}^{+im*}}, \psi_{C_r}^*e^{i2\pi\psi_{C_r}^{+im*}}, r_{C_r}^*e^{i2\pi r_{C_r}^{+im*}}\right)$ are two collections of C_rC-PiF values, where $r = 1,2,...,m$. If $\eta_{C_r} \leq \eta_{C_r}^*$, $\eta_{C_r}^{im} \leq \eta_{C_r}^{im*}$, $\phi_{C_r} \geq \phi_{C_r}^*$, $\phi_{C_r}^{im} \geq \phi_{C_r}^{im*}$, $\psi_{C_r} \geq \psi_{C_r}^*$, $\psi_{C_r}^{im} \geq \psi_{C_r}^{im*}$, $r_{C_r} \leq r_{C_r}^*$, and $r_{C_r}^{im} \leq r_{C_r}^{im*}$ then $\left(\eta_{C_r}\right) \leq \left(\eta_{C_r}^*\right)$. As we have

$$\begin{aligned}\left(\eta_{C_r}\right) &\leq \left(\eta_{C_r}^*\right)\\ &\Rightarrow \alpha_r\left(\eta_{C_r}\right) \leq \alpha_r\left(\eta_{C_r}^*\right)\\ &\Rightarrow \left(\alpha_r\left(\eta_{C_r}\right)\right)^\lambda \leq \left(\alpha_r\left(\eta_{C_r}^*\right)\right)^\lambda\\ &\Rightarrow \sum_{r=1}^m\left(\alpha_r\left(\eta_{C_r}\right)\right)^\lambda \leq \left(\alpha_r\left(\eta_{C_r}^*\right)\right)^\lambda\end{aligned}$$

$$\begin{aligned}&\Rightarrow \left(\sum_{r=1}^m\left(\alpha_r\left(\eta_{C_r}\right)\right)^\lambda\right)^{\frac{1}{\lambda}} \leq \left(\sum_{r=1}^m\left(\alpha_r\left(\eta_{C_r}^*\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\\ &\Rightarrow \left(\left(\sum_{r=1}^m\left(\alpha_r\left(\eta_{C_r}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right) \leq \left(\left(\sum_{r=1}^m\left(\alpha_r\left(\eta_{C_r}^*\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right).\end{aligned}$$

Thus

$$\left(\left(\sum_{r=1}^m\left(\alpha_r\left(\eta_{C_r}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right) \leq \left(\left(\sum_{r=1}^m\left(\alpha_r\left(\eta_{C_r}^*\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right). \qquad \text{(i)}$$

Further,

$$\begin{aligned}\left(\eta_{C_r}^{im}(u)\right) &\leq \left(\eta_{C_r}^{im*}(u)\right) \leq \\ &\Rightarrow \alpha_r\left(\eta_{C_r}^{im}(u)\right) \leq \alpha_r\left(\eta_{C_r}^{im*}(u)\right) \\ &\Rightarrow \left(\alpha_r\left(\eta_{C_r}^{im}(u)\right)\right)^{\lambda} \leq \left(\alpha_r\left(\eta_{C_r}^{im*}(u)\right)\right)^{\lambda} \\ &\Rightarrow \sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{im}(u)\right)\right)^{\lambda} \leq \left(\alpha_r\left(\eta_{C_r}^{im*}(u)\right)\right)^{\lambda}\end{aligned}$$

$$\begin{aligned}&\Rightarrow \left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}} \leq \left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{im*}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}} \\ &\Rightarrow \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right) \leq \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{im*}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).\end{aligned}$$

Hence

$$e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{+im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)} \leq e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{+im*}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}. \qquad \text{(ii)}$$

From (i) and (ii)

$$\begin{aligned}&\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{+im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)} \\ &\leq \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\eta_{C_r}^{+im*}\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}. \qquad (A)\end{aligned}$$

Now

$$\begin{aligned}\left(\phi_{C_r}\right) &\geq \left(\phi_{C_r}^{*}\right) \\ &\Rightarrow -\left(\phi_{C_r}\right) \leq -\left(\phi_{C_r}^{*}\right) \\ &\Rightarrow \left(1-\left(\phi_{C_r}\right)\right) \leq \left(1-\left(\phi_{C_r}^{*}\right)\right) \\ &\Rightarrow \alpha_r\left(1-\left(\phi_{C_r}\right)\right) \leq \alpha_r\left(1-\left(\phi_{C_r}^{*}\right)\right) \\ &\Rightarrow \left(\alpha_r\left(1-\left(\phi_{C_r}\right)\right)\right)^{\lambda} \leq \left(\alpha_r\left(1-\left(\phi_{C_r}^{*}\right)\right)\right)^{\lambda}.\end{aligned}$$

Then we have

$$
\begin{aligned}
& \sum_{r=1}^{m} \left(\alpha_r \left(1-\left(\phi_{C_r}\right)\right)\right)^{\lambda} \\
\leq \; & \sum_{r=1}^{m} \left(\alpha_r \left(1-\left(\phi^{*}_{C_r}\right)\right)\right)^{\lambda} \\
\Rightarrow \; & -\sum_{r=1}^{m} \left(\alpha_r \left(1-\left(\phi_{C_r}\right)\right)\right)^{\lambda} \geq -\sum_{r=1}^{m} \left(\alpha_r \left(1-\left(\phi^{*}_{C_r}\right)\right)\right)^{\lambda} \\
\Rightarrow \; & 1-\sum_{r=1}^{m} \left(\alpha_r \left(1-\left(\phi_{C_r}\right)\right)\right)^{\lambda} \geq 1-\sum_{r=1}^{m} \left(\alpha_r \left(1-\left(\phi^{*}_{C_r}\right)\right)\right)^{\lambda} \\
\Rightarrow \; & \left(1-\sum_{r=1}^{m} \left(\alpha_r \left(1-\left(\phi_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}} \geq \left(1-\sum_{r=1}^{m} \left(\alpha_r \left(1-\left(\phi^{*}_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}} \\
\Rightarrow \; & \left(\left(1-\sum_{r=1}^{m} \left(\alpha_r \left(1-\left(\phi_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right) \geq \left(\left(1-\sum_{r=1}^{m} \left(\alpha_r \left(1-\left(\phi^{*}_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right). \quad \text{(iii)}
\end{aligned}
$$

Also

$$
\begin{aligned}
& \sum_{r=1}^{m} \left(\alpha_r \left(1-\left(\phi^{im}_{C_r}\right)\right)\right)^{\lambda} \\
\leq \; & \sum_{r=1}^{m} \left(\alpha_r \left(1-\left(\phi^{im*}_{C_r}\right)\right)\right)^{\lambda} \\
\Rightarrow \; & -\sum_{r=1}^{m} \left(\alpha_r \left(1-\left(\phi^{im}_{C_r}\right)\right)\right)^{\lambda} \geq -\sum_{r=1}^{m} \left(\alpha_r \left(1-\left(\phi^{im*}_{C_r}\right)\right)\right)^{\lambda} \\
\Rightarrow \; & 1-\sum_{r=1}^{m} \left(\alpha_r \left(1-\left(\phi^{im}_{C_r}\right)\right)\right)^{\lambda} \geq 1-\sum_{r=1}^{m} \left(\alpha_r \left(1-\left(\phi^{im*}_{C_r}\right)\right)\right)^{\lambda}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \; & \left(1-\sum_{r=1}^{m} \left(\alpha_r \left(1-\left(\phi^{im}_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}} \geq \left(1-\sum_{r=1}^{m} \left(\alpha_r \left(1-\left(\phi^{im*}_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}} \\
\Rightarrow \; & \left(\left(1-\sum_{r=1}^{m} \left(\alpha_r \left(1-\left(\phi^{im}_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right) \geq \left(\left(1-\sum_{r=1}^{m} \left(\alpha_r \left(1-\left(\phi^{im*}_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).
\end{aligned}
$$

This implies that

$$
e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi^{im}_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]} \geq e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi^{im*}_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}. \quad \text{(iv)}
$$

From (iii) and (iv) , we get

$$
\left(\left(1-\sum_{r=1}^{m} \left(\alpha_r \left(1-\left(\phi^{im}_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right) .e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1\left(\phi^{+im}_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}
$$

$$\geq \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im}\right) \right) \right) ^{\lambda }\right) ^{\frac{1}{\lambda }}\right) .e^{i2\pi \left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{+im*}\right) \right) \right) ^{\lambda }\right) ^{\frac{1}{\lambda }}\right) \right] }. \qquad (B).$$

Now

$$\begin{aligned} \left(\psi_{C_r}\right) &\geq \left(\psi_{C_r}^{*}\right) \\ &\Rightarrow -\left(\psi_{C_r}\right) \leq -\left(\psi_{C_r}^{*}\right) \\ &\Rightarrow \left(1-\left(\psi_{C_r}\right) \right) \leq \left(1-\left(\psi_{C_r}^{*}\right) \right) \\ &\Rightarrow \alpha_r\left(1-\left(\psi_{C_r}\right) \right) \leq \alpha_r\left(1-\left(\psi_{C_r}^{*}\right) \right) \\ &\Rightarrow \left(\alpha_r\left(1-\left(\psi_{C_r}\right) \right) \right) ^{\lambda }\leq \left(\alpha_r\left(1-\left(\psi_{C_r}^{*}\right) \right) \right) ^{\lambda }. \end{aligned}$$

Then we have

$$\begin{aligned} &\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}\right) \right) \right) ^{\lambda } \\ \leq\ &\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{*}\right) \right) \right) ^{\lambda } \\ \Rightarrow\ &-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}\right) \right) \right) ^{\lambda }\geq -\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{*}\right) \right) \right) ^{\lambda } \\ \Rightarrow\ &1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}\right) \right) \right) ^{\lambda }\geq 1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{*}\right) \right) \right) ^{\lambda } \\ \Rightarrow\ &\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}\right) \right) \right) ^{\lambda }\right) ^{\frac{1}{\lambda }}\geq \left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{*}\right) \right) \right) ^{\lambda }\right) ^{\frac{1}{\lambda }} \\ \Rightarrow\ &\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}\right) \right) \right) ^{\lambda }\right) ^{\frac{1}{\lambda }}\right) \geq \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{*}\right) \right) \right) ^{\lambda }\right) ^{\frac{1}{\lambda }}\right) . \qquad \text{(v)} \end{aligned}$$

Also

$$\begin{aligned} &\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im}\right) \right) \right) ^{\lambda } \\ \leq\ &\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im*}\right) \right) \right) ^{\lambda } \\ \Rightarrow\ &-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im}\right) \right) \right) ^{\lambda }\geq -\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im*}\right) \right) \right) ^{\lambda } \\ \Rightarrow\ &1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im}\right) \right) \right) ^{\lambda }\geq 1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im*}\right) \right) \right) ^{\lambda } \end{aligned}$$

$$\begin{aligned} \Rightarrow\ &\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im}\right) \right) \right) ^{\lambda }\right) ^{\frac{1}{\lambda }}\geq \left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im*}\right) \right) \right) ^{\lambda }\right) ^{\frac{1}{\lambda }} \\ \Rightarrow\ &\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im}\right) \right) \right) ^{\lambda }\right) ^{\frac{1}{\lambda }}\right) \geq \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im*}\right) \right) \right) ^{\lambda }\right) ^{\frac{1}{\lambda }}\right) . \end{aligned}$$

This implies that

$$e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]} \geq e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im*}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}. \qquad \text{(vi)}$$

From (v) and (vi) , we get

$$\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{+im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}$$

$$\geq \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{+im*}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}. \qquad (C).$$

Now

$$\begin{aligned}
\left(r_{C_r}\right) &\leq \left(r_{C_r}^{*}\right)\\
&\Rightarrow \alpha_r\left(r_{C_r}\right) \leq \alpha_r\left(r_{C_r}^{*}\right)\\
&\Rightarrow \left(\alpha_r\left(r_{C_r}\right)\right)^{\lambda} \leq \left(\alpha_r\left(r_{C_r}^{*}\right)\right)^{\lambda}\\
&\Rightarrow \sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}\right)\right)^{\lambda} \leq \left(\alpha_r\left(r_{C_r}^{*}\right)\right)^{\lambda}
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}} \leq \left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{*}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\\
&\Rightarrow \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right) \leq \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{*}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).
\end{aligned}$$

Thus

$$\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right) \leq \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{*}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right). \qquad \text{(vii)}$$

Further,

$$\begin{aligned}
\left(r_{C_r}^{im}(u)\right) &\leq \left(r_{C_r}^{im*}(u)\right) \leq\\
&\Rightarrow \alpha_r\left(r_{C_r}^{im}(u)\right) \leq \alpha_r\left(r_{C_r}^{im*}(u)\right)\\
&\Rightarrow \left(\alpha_r\left(r_{C_r}^{im}(u)\right)\right)^{\lambda} \leq \left(\alpha_r\left(r_{C_r}^{im*}(u)\right)\right)^{\lambda}\\
&\Rightarrow \sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{im}(u)\right)\right)^{\lambda} \leq \left(\alpha_r\left(r_{C_r}^{im*}(u)\right)\right)^{\lambda}
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}} \leq \left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{im*}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\\
&\Rightarrow \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right) \leq \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{im*}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).
\end{aligned}$$

Hence

$$e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{+im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)} \leq e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{+im*}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}. \qquad \text{(viii)}$$

From (i) and (ii)

$$\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{+im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}$$

$$\leq \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{*}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{+im*}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}. \qquad (D)$$

Thus, from (A) ,(B) ,(C) and (D) , we have

$$\begin{pmatrix} \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{+im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1\left(\phi_{C_r}^{+im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{+im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{+im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)} \end{pmatrix}$$

$$\leq \begin{pmatrix} \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{*}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{+im*}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{+im*}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\psi_{C_r}^{+im*}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{*}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{+im*}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]} \end{pmatrix}.$$

Hence $CrC\text{-}PiFWAM_E\left(P_1, P_2, P_3, ..., P_m\right) \leq CrC\text{-}PiFWAM_E\left(P_1^*, P_2^*, P_3^*, ..., P_m^*\right)$.

Let $\left\{P_r = \left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}}, \psi_{C_r}e^{i2\pi\psi_{C_r}^{+im}}, r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ and $\left\{P_r^* = \left(\eta_{C_r}^*e^{i2\pi\eta_{C_r}^{+im*}}, \phi_{C_r}^*e^{i2\pi\phi_{C_r}^{+im*}}, \psi_{C_r}^*e^{i2\pi\psi_{C_r}^{+im*}}, r_{C_r}^*e^{i2\pi r_{C_r}^{+im*}}\right) : r = 1, 2, ..., m\right\}$ are two collections of $C_rC\text{-}PiF$ values. If $\eta_{C_r} \leq \eta_{C_r}^*$, $\eta_{C_r}^{im} \leq \eta_{C_r}^{im*}$, $\phi_{C_r} \geq \phi_{C_r}^*$, $\phi_{C_r}^{im} \geq \phi_{C_r}^{im*}$, $\psi_{C_r} \geq \psi_{C_r}^*$, $\psi_{C_r}^{im} \geq \psi_{C_r}^{im*}$, $r_{C_r} \geq r_{C_r}^*$, and $r_{C_r}^{im} \geq r_{C_r}^{im*}$ where $r = 1, 2, ..., m$, then $C_r^2C\text{-}PiFWAM_E\left(P_1, P_2, P_3, ..., P_m\right) \leq C_r^2C\text{-}PiFWAM_E\left(P_1^*, P_2^*, P_3^*, ..., P_m^*\right)$.

Similar to the proof of Theorem 2.2.

Let $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, \psi_{C_r} e^{i2\pi\psi^{+im}_{C_r}}, r_{C_r} e^{i2\pi r^{+im}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ be the collection of C_rC-PiF values and let C^1_rC-$PiFOWAM : \Omega^m \to \Omega$,

if C^1_rC-$PiFOWAM_E\left(P_1, P_2, P_3, ..., P_m\right) = \left(\left(\alpha_1 P_{\delta(1)}\right)^\lambda \oplus \left(\alpha_2 P_{\delta(1)}\right)^\lambda \oplus \left(\alpha_3 P_{\delta(3)}\right)^\lambda \oplus ... \oplus \left(\alpha_m P_{\delta(m)}\right)^\lambda\right)^{\frac{1}{\lambda}}$ then C^1_rC-$PiFOWAM$ is called a Circular complex Picture fuzzy ordered weighted averaging mean operator of dimension n, where $\left(\delta\left(1\right), \delta\left(2\right), ..., \delta\left(m\right)\right)$ is a permutation of $\left(1, 2, ..., m\right)$ such that $P_{\delta(r-1)} \geq P_{\delta(r)}$ for all r, Ω is the set of all C_rC-PiF values, $E = \left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m\right)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\Sigma^m_{r=1}\alpha_r = 1$, where $r = 1, 2, ..., m$.

Let $P_r = \left(\eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, \psi_{C_r} e^{i2\pi\psi^{+im}_{C_r}}, r_{C_r} e^{i2\pi r^{+im}_{C_r}}\right)$ be the collection of CrC-PiF values, where $r = 1, 2, ..., m$. Then by using the C_rC-$PiFOWAM_E$ operator their aggregated value is also a C^1_rC-PiF value and

$$C^1_rC - PiFOWAM_E\left(P_1, P_2, P_3, ..., P_m\right)$$

$$= \begin{pmatrix} \left(\left(\sum\limits_{r=1}^m \left(\alpha_r \left(\eta_{C_{\delta(r)}}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right) .e^{i2\pi\left[\left(\left(\sum\limits_{r=1}^m \left(\alpha_r\left(\eta^{+im}_{C_{\delta(r)}}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left(\left(1 - \sum\limits_{r=1}^m \left(\alpha_r \left(1 - \left(\phi_{C_{\delta(r)}}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right) .e^{i2\pi\left(\left(1-\sum\limits_{r=1}^m \left(\alpha_r\left(1-\left(\phi^{+im}_{C_{\delta(r)}}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1 - \sum\limits_{r=1}^m \left(\alpha_r \left(1 - \left(\psi_{C_{\delta(r)}}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right) .e^{i2\pi\left(\left(1-\sum\limits_{r=1}^m \left(\alpha_r\left(1-\left(\psi^{+im}_{C_{\delta(r)}}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(\sum\limits_{r=1}^m \left(\alpha_r \left(r_{C_{\delta(r)}}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right) .e^{i2\pi\left[\left(\left(\sum\limits_{r=1}^m \left(\alpha_r\left(r^{+im}_{C_{\delta(r)}}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)\right]}, \end{pmatrix}$$

where $E = \left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_n\right)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\Sigma^n_{r=1}\alpha_r = 1$, where .

Similar to the proof of Theorem 2.2.

Let $P_r = \left(\eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, \psi_{C_r} e^{i2\pi\psi^{+im}_{C_r}}, r_{C_r} e^{i2\pi r^{+im}_{C_r}}\right)$ be the collection of CrC-PiF values, where $r = 1, 2, ..., m$. Then C^2_rCT-$PiFOWAM_E$ operator is defined by

$$C^2_rCT - SPFOWAM_E\left(P_1, P_2, P_3, ..., P_m\right)$$

$$= \begin{pmatrix} \left(\left(\sum\limits_{r=1}^m \left(\alpha_r \left(\eta_{C_{\delta(r)}}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right) .e^{i2\pi\left[\left(\left(\sum\limits_{r=1}^m \left(\alpha_r\left(\eta^{+im}_{C_{\delta(r)}}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left(\left(1 - \sum\limits_{r=1}^m \left(\alpha_r \left(1 - \left(\phi_{C_{\delta(r)}}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right) .e^{i2\pi\left(\left(1-\sum\limits_{r=1}^m \left(\alpha_r\left(1-\left(\phi^{+im}_{C_{\delta(r)}}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1 - \sum\limits_{r=1}^m \left(\alpha_r \left(1 - \left(\psi_{C_{\delta(r)}}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right) .e^{i2\pi\left(\left(1-\sum\limits_{r=1}^m \left(\alpha_r\left(1-\left(\psi^{+im}_{C_{\delta(r)}}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1 - \sum\limits_{r=1}^m \left(\alpha_r \left(1 - \left(r_{C_{\delta(r)}}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right) .e^{i2\pi\left(\left(1-\sum\limits_{r=1}^m \left(\alpha_r\left(1-\left(r^{+im}_{C_{\delta(r)}}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)},, \end{pmatrix}$$

where $E = \left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_n\right)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\Sigma^n_{r=1}\alpha_r = 1$, where .

Similar to the proof of Theorem 2.2.

Let $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}},\phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}},\psi_{C_r}e^{i2\pi\psi_{C_r}^{+im}},r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right):r=1,2,...,m\right\}$ be the collection of C_rC-PiF values. Let $E=(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r\in[0,1]$ and $\Sigma_{r=1}^{m}\alpha_r=1$. If $\left(\eta_{C_1}.e^{i2\pi\left[\eta_{C_1}^{+im}\right]},\phi_{C_1}e^{i2\pi\left[\psi_{C_1}^{+im}\right]},\psi_{C_1}e^{i2\pi\left[\psi_{C_1}^{+im}\right]},r_{C_1}.e^{i2\pi\left[\eta_{C_1}^{+im}\right]}\right)=\left(\eta_{C_2}.e^{i2\pi\left[\eta_{C_2}^{+im}\right]},\phi_{C_2}e^{i2\pi\left[\psi_{C_2}^{+im}\right]},\psi_{C_2}e\right.$
...

$=\left(\eta_{C_m}.e^{i2\pi\eta_{C_m}^{+im}},\phi_{C_m}e^{i2\pi\psi_{C_m}^{+im}},\psi_{C_m}e^{i2\pi\psi_{C_m}^{+im}},r_{C_m}.e^{i2\pi\eta_{C_m}^{+im}}\right)$

$=\left(\eta_C.e^{i2\pi\eta_C^{+im}},\phi_Ce^{i2\pi\psi_C^{+im}},\psi_Ce^{i2\pi\psi_C^{+im}},r_C.e^{i2\pi\eta_C^{+im}}\right)$ and $\lambda=1$, then

$C_r^1C\text{-}PiFOWAM_E\left(P_1,P_2,P_3,...,P_m\right)=\left(\eta_C.e^{i2\pi\eta_C^{+im}},\phi_Ce^{i2\pi\psi_C^{+im}},\psi_Ce^{i2\pi\psi_C^{+im}},r_C.e^{i2\pi\eta_C^{+im}}\right).$

Similar to the proof of Theorem 2.2.

Let $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}},\phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}},\psi_{C_r}e^{i2\pi\psi_{C_r}^{+im}},r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right):r=1,2,...,m\right\}$ be the collection of C_rC-PiF values. Let $E=(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r\in[0,1]$ and $\Sigma_{r=1}^{m}\alpha_r=1$. If $\left(\eta_{C_1}.e^{i2\pi\left[\eta_{C_1}^{+im}\right]},\phi_{C_1}e^{i2\pi\left[\psi_{C_1}^{+im}\right]},\psi_{C_1}e^{i2\pi\left[\psi_{C_1}^{+im}\right]},r_{C_1}.e^{i2\pi\left[\eta_{C_1}^{+im}\right]}\right)=\left(\eta_{C_2}.e^{i2\pi\left[\eta_{C_2}^{+im}\right]},\phi_{C_2}e^{i2\pi\left[\psi_{C_2}^{+im}\right]},\psi_{C_2}e\right.$
...

$=\left(\eta_{C_m}.e^{i2\pi\eta_{C_m}^{+im}},\phi_{C_m}e^{i2\pi\psi_{C_m}^{+im}},\psi_{C_m}e^{i2\pi\psi_{C_m}^{+im}},r_{C_m}.e^{i2\pi\eta_{C_m}^{+im}}\right)$

$=\left(\eta_C.e^{i2\pi\eta_C^{+im}},\phi_Ce^{i2\pi\psi_C^{+im}},\psi_Ce^{i2\pi\psi_C^{+im}},r_C.e^{i2\pi\eta_C^{+im}}\right)$ and $\lambda=1$, then

$C_r^2C\text{-}PiFOWAM_E\left(P_1,P_2,P_3,...,P_m\right)=\left(\eta_C.e^{i2\pi\eta_C^{+im}},\phi_Ce^{i2\pi\psi_C^{+im}},\psi_Ce^{i2\pi\psi_C^{+im}},r_C.e^{i2\pi\eta_C^{+im}}\right).$

Similar to the proof of Theorem 2.2.

Let $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}},\phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}},\psi_{C_r}e^{i2\pi\psi_{C_r}^{+im}},r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right):r=1,2,...,m\right\}$ be the collection of C_rC-PiF values and $E=(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r\in[0,1]$ and $\Sigma_{r=1}^{m}\alpha_r=1$. Let

$$P^-=\left(\min_{1\le r\le m}\eta_{C_r}.e^{i2\pi\min\limits_{1\le r\le m}\eta_{C_r}^{im}},\max_{1\le r\le m}\phi_{C_r}.e^{i2\pi\max\limits_{1\le r\le m}\phi_{C_r}^{im}},\max_{1\le r\le m}\psi_{C_r}.e^{i2\pi\max\limits_{1\le r\le m}\psi_{C_r}^{im}},\min_{1\le r\le m}r_{C_r}.e^{i2\pi\min\limits_{1\le r\le m}r_{C_r}^{im}}\right)$$

and

$$P^+=\left(\max_{1\le r\le m}\eta_{C_r}.e^{i2\pi\max\limits_{1\le r\le m}\eta_{C_r}^{im}},\min_{1\le r\le m}\phi_{C_r}.e^{i2\pi\min\limits_{1\le r\le m}\phi_{C_r}^{im}},\min_{1\le r\le m}\psi_{C_r}.e^{i2\pi\min\limits_{1\le r\le m}\psi_{C_r}^{im}},\max_{1\le r\le m}r_{C_r}.e^{i2\pi\max\limits_{1\le r\le m}r_{C_r}^{im}}\right),$$

where and $\lambda=1$. Then $P^-\le C_r^1C\text{-}PiFOWAM_E\left(P_1,P_2,P_3,...,P_m\right)\le P^+$.

Similar to the proof of Theorem 2.2.

Let $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}},\phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}},\psi_{C_r}e^{i2\pi\psi_{C_r}^{+im}},r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right):r=1,2,...,m\right\}$ be the collection of C_rC-PiF values and $E=(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r\in[0,1]$ and $\Sigma_{r=1}^{m}\alpha_r=1$. Let

$$P^-=\left(\min_{1\le r\le m}\eta_{C_r}.e^{i2\pi\min\limits_{1\le r\le m}\eta_{C_r}^{im}},\max_{1\le r\le m}\phi_{C_r}.e^{i2\pi\max\limits_{1\le r\le m}\phi_{C_r}^{im}},\max_{1\le r\le m}\psi_{C_r}.e^{i2\pi\max\limits_{1\le r\le m}\psi_{C_r}^{im}},\max_{1\le r\le m}r_{C_r}.e^{i2\pi\max\limits_{1\le r\le m}r_{C_r}^{im}}\right)$$

and

$$P^+=\left(\max_{1\le r\le m}\eta_{C_r}.e^{i2\pi\max\limits_{1\le r\le m}\eta_{C_r}^{im}},\min_{1\le r\le m}\phi_{C_r}.e^{i2\pi\min\limits_{1\le r\le m}\phi_{C_r}^{im}},\min_{1\le r\le m}\psi_{C_r}.e^{i2\pi\min\limits_{1\le r\le m}\psi_{C_r}^{im}},\min_{1\le r\le m}r_{C_r}.e^{i2\pi\min\limits_{1\le r\le m}r_{C_r}^{im}}\right),$$

where and $\lambda=1$. Then $P^-\le C_r^2C\text{-}PiFOWAM_E\left(P_1,P_2,P_3,...,P_m\right)\le P^+$.

Similar to the proof of Theorem 2.2.

Let $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}},\phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}},\psi_{C_r}e^{i2\pi\psi_{C_r}^{+im}},r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right):r=1,2,...,m\right\}$ and $\left\{P_r^*=\left(\eta_{C_r}^*e^{i2\pi\eta_{C_r}^{+im*}},\phi_{C_r}^*e^{i2\pi\phi_{C_r}^{+im*}},\psi_{C_r}^*e^{i2\pi\psi_{C_r}^{+im*}},r_{C_r}^*e^{i2\pi r_{C_r}^{+im*}}\right):r=1,2,...,m\right\}$ are two collections of $C_rCT\text{-}SF$ values. If $\eta_{C_r}\le\eta_{C_r}^*$, $\eta_{C_r}^{im}\le\eta_{C_r}^{im*}$, $\phi_{C_r}\ge\phi_{C_r}^*$, $\phi_{C_r}^{im}\ge\phi_{C_r}^{im*}$, $\psi_{C_r}\ge\psi_{C_r}^*$, $\psi_{C_r}^{im}\ge\psi_{C_r}^{im*}$, $r_{C_r}\le r_{C_r}^*$, and $r_{C_r}^{im}\le r_{C_r}^{im*}$ where $r=1,2,...,m$, then $C_r^1C\text{-}PiFOWAM_E\left(P_1,P_2,P_3,...,P_m\right)\le C_r^1C\text{-}PiFOWAM_E\left(P_1^*,P_2^*,P_3^*,...,P_m^*\right)$

Similar to the proof of Theorem 2.2.

Let $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, \psi_{C_r} e^{i2\pi\psi^{+im}_{C_r}}, r_{C_r} e^{i2\pi r^{+im}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ and $\left\{P^*_r = \left(\eta^*_{C_r} e^{i2\pi\eta^{+im*}_{C_r}}, \phi^*_{C_r} e^{i2\pi\phi^{+im*}_{C_r}}, \psi^*_{C_r} e^{i2\pi\psi^{+im*}_{C_r}}, r^*_{C_r} e^{i2\pi r^{+im*}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ are two collections of C_rCT-SPF values. If $\eta_{C_r} \leq \eta^*_{C_r}$, $\eta^{im}_{C_r} \leq \eta^{im*}_{C_r}$, $\phi_{C_r} \geq \phi^*_{C_r}$, $\phi^{im}_{C_r} \geq \phi^{im*}_{C_r}$, $\psi_{C_r} \geq \psi^*_{C_r}$, $\psi^{im}_{C_r} \geq \psi^{im*}_{C_r}$, $r_{C_r} \geq r^*_{C_r}$, and $r^{im}_{C_r} \geq r^{im*}_{C_r}$ where $r = 1, 2, ..., m,$

then C^2_rC-$PiFOWAM_E\left(P_1, P_2, P_3, ..., P_m\right) \leq C^2_rC$-$PiFOWAM_E\left(P^*_1, P^*_2, P^*_3, ..., P^*_m\right)$

Similar to the proof of Theorem 2.2.

3. Complex circular complex picture fuzzy weighted geometric mean aggregation operators

Here, we provide some new geometric aggregation operators, complex C_rC-PiF weighted geometric mean aggregation operator (C_rC-PiPFWGM) and C_rC-PiF ordered weighted geometric mean aggregation operator (C_rC-$PiFOWGM$), using the suggested operations.

Let $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, \psi_{C_r} e^{i2\pi\psi^{+im}_{C_r}}, r_{C_r} e^{i2\pi r^{+im}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ be the collection of C_rC-PiF values and let C_rC-PiFOWGM: $\Omega^m \rightarrow \Omega$, if

C_rC-PiFWGM$_E\left(P_1, P_2, P_3, ..., P_m\right) = \left(\left(P_1^{\odot\alpha_1}\right)^\lambda \otimes \left(P_2^{\odot\alpha_2}\right)^\lambda \otimes \left(P_3^{\odot\alpha_3}\right)^\lambda \otimes ... \otimes \left(P_m^{\odot\alpha_m}\right)^\lambda\right)^{\frac{1}{\lambda}}$ then C_rC-$PiFWGM$ is called a Circular complex picture fuzzy weighted geometric mean operator of dimension n, where Ω is the set of all C_rC-PiF values, $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\Sigma^m_{r=1}\alpha_r = 1$, where $r = 1, 2, ..., m.$

Let $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, \psi_{C_r} e^{i2\pi\psi^{+im}_{C_r}}, r_{C_r} e^{i2\pi r^{+im}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ be the collection of C_rCT-SF values. Then by using the C_rCT-$SFWGM_E$ operator their aggregated value is also a C_rCT-SF value and

$$
\begin{aligned}
&= C^1_rC\text{-PiFWGM}_E\left(P_1, P_2, P_3, ..., P_m\right) \\
&= \left(\begin{array}{c}
\left(\left(1 - \sum_{r=1}^{m}\left(\alpha_r\left(1 - \left(\eta_{C_r}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\eta^{+im}_{C_r}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)\right]}, \\
\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\phi_{C_r}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)\right].e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\phi^{+im}_{C_r}(u)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)\right]}, \\
\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\psi_{C_r}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)\right].e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\psi^{+im}_{C_r}(u)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)\right]}, \\
\left(\left(1 - \sum_{r=1}^{m}\left(\alpha_r\left(1 - \left(r_{C_r}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(r^{+im}_{C_r}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)\right]}
\end{array}\right),
\end{aligned}
$$

where $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_n)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\Sigma^n_{r=1}\alpha_r = 1$, where $r = 1, 2, ..., m.$

Similar to the proof of Theorem 2.2.

Let $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, \psi_{C_r} e^{i2\pi\psi^{+im}_{C_r}}, r_{C_r} e^{i2\pi r^{+im}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ be the collection of C_rC-PiF values. Then by using the C_rC-$PiFWGM_E$ operator their aggregated value is also a

$C_rC\text{-}PiF$ value and

$$= C_r^2C\text{-PiFWGM}_E\left(P_1, P_2, P_3, ..., P_m\right)$$

$$= \left(\begin{array}{c} \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\eta_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\eta_{C_r}^{+im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\phi_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right].e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\phi_{C_r}^{+im}(u)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\psi_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right].e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\psi_{C_r}^{+im}(u)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right].e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{+im}(u)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]} \end{array}\right),$$

where $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_n)^T$ are a weight vectors of P_r with $\alpha_r \in [0,1]$ and $\Sigma_{r=1}^{n}\alpha_r = 1$, where $r = 1, 2, ..., m$.

Similar to the proof of Theorem 2.2.

Let $\left\{P_r = \left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}}, \psi_{C_r}e^{i2\pi\psi_{C_r}^{+im}}, r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ be the collection of $C_rC\text{-}PiF$ values. Let $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0,1]$ and $\Sigma_{r=1}^{m}\alpha_r = 1$. If $\left(\eta_{C_1}e^{i2\pi\eta_{C1}^{+im}}, \phi_{C_1}e^{i2\pi\phi_{C_1}^{+im}}, \psi_{C_1}e^{i2\pi\psi_{C_1}^{+im}}, r_{C_1}e^{i2\pi r_{C_1}^{+im}}\right) =$
$\left(\eta_{C_2}e^{i2\pi\eta_{C_2}^{+im}}, \phi_{C_2}e^{i2\pi\phi_{C_2}^{+im}}, \psi_{C_2}e^{i2\pi\psi_{C_2}^{+im}}, r_{C_2}e^{i2\pi r_{C_2}^{+im}}\right) = ...$
$= \left(\eta_{C_m}e^{i2\pi\eta_{Cm}^{+im}}, \phi_{C_m}e^{i2\pi\phi_{Cm}^{+im}}, \psi_{C_m}e^{i2\pi\psi_{Cm}^{+im}}, r_{C_m}e^{i2\pi r_{Cm}^{+im}}\right)$
$= \left(\eta_{C}e^{i2\pi\eta_{C}^{+im}}, \phi_{C}e^{i2\pi\phi_{C}^{+im}}, \psi_{C}e^{i2\pi\psi_{C}^{+im}}, r_{C}e^{i2\pi r_{C}^{+im}}\right)$ and $\lambda = 1$, then
$C_r^1C\text{-PiFWGM}_E\left(P_1, P_2, P_3, ..., P_m\right) = \left(\eta_{C}e^{i2\pi\eta_{C}^{+im}}, \phi_{C}e^{i2\pi\phi_{C}^{+im}}, \psi_{C}e^{i2\pi\psi_{C}^{+im}}, r_{C}e^{i2\pi r_{C}^{+im}}\right)$.

Similar to the proof of Theorem 2.2.

Let $\left\{P_r = \left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}}, \psi_{C_r}e^{i2\pi\psi_{C_r}^{+im}}, r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ be the collection of C_rC-PiF values. Let $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0,1]$ and $\Sigma_{r=1}^{m}\alpha_r = 1$. If $\left(\eta_{C_1}e^{i2\pi\eta_{C1}^{+im}}, \phi_{C_1}e^{i2\pi\phi_{C_1}^{+im}}, \psi_{C_1}e^{i2\pi\psi_{C_1}^{+im}}, r_{C_1}e^{i2\pi r_{C_1}^{+im}}\right) =$
$\left(\eta_{C_2}e^{i2\pi\eta_{C_2}^{+im}}, \phi_{C_2}e^{i2\pi\phi_{C_2}^{+im}}, \psi_{C_2}e^{i2\pi\psi_{C_2}^{+im}}, r_{C_2}e^{i2\pi r_{C_2}^{+im}}\right) = ...$
$= \left(\eta_{C_m}e^{i2\pi\eta_{Cm}^{+im}}, \phi_{C_m}e^{i2\pi\phi_{Cm}^{+im}}, \psi_{C_m}e^{i2\pi\psi_{Cm}^{+im}}, r_{C_m}e^{i2\pi r_{Cm}^{+im}}\right)$
$= \left(\eta_{C}e^{i2\pi\eta_{C}^{+im}}, \phi_{C}e^{i2\pi\phi_{C}^{+im}}, \psi_{C}e^{i2\pi\psi_{C}^{+im}}, r_{C}e^{i2\pi r_{C}^{+im}}\right)$ and $\lambda = 1$, then
$C_r^2C\text{-PiFWGM}_E\left(P_1, P_2, P_3, ..., P_m\right) = \left(\eta_{C}e^{i2\pi\eta_{C}^{+im}}, \phi_{C}e^{i2\pi\phi_{C}^{+im}}, \psi_{C}e^{i2\pi\psi_{C}^{+im}}, r_{C}e^{i2\pi r_{C}^{+im}}\right)$.

Similar to the proof of Theorem 2.2.

Let $\left\{P_r = \left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}}, \psi_{C_r}e^{i2\pi\psi_{C_r}^{+im}}, r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ be the collection of C_rC-PiF values and $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0,1]$ and $\Sigma_{r=1}^{m}\alpha_r = 1$.

Let
$$P^{-}=\left(\min_{1\le r\le m}\eta_{C_r}.e^{i2\pi\left[\min_{1\le r\le m}\eta_{C_r}^{+im}\right]},\max_{1\le r\le m}\psi_{C_r}^{-}.e^{i2\pi\left[\max_{1\le r\le m}\psi_{C_r}^{+im}\right]},\max_{1\le r\le m}\psi_{C_r}^{-}.e^{i2\pi\left[\max_{1\le r\le m}\psi_{C_r}^{+im}\right]},\min_{1\le r\le m}\eta_{C_r}.e^{i2\pi\left[\min_{1\le r\le m}\right.}\right.$$
and
$$P^{+}=\left(\max_{1\le r\le m}\eta_{C_r}.e^{i2\pi\left[\max_{1\le r\le m}\eta_{C_r}^{+im}\right]},\min_{1\le r\le m}\psi_{C_r}.e^{i2\pi\left[\min_{1\le r\le m}\psi_{C_r}^{+im}\right]},\min_{1\le r\le m}\psi_{C_r}.e^{i2\pi\left[\min_{1\le r\le m}\psi_{C_r}^{+im}\right]},\max_{1\le r\le m}\eta_{C_r}.e^{i2\pi\left[\max_{1\le r\le m}\right.}\right.$$
where and $\lambda=1$. Then $P^{-}\le C_r^1C$-PiFWGM$_E\left(P_1,P_2,P_3,...,P_m\right)\le P^{+}$.

Similar to the proof of Theorem 2.2.

Let $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}},\phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}},\psi_{C_r}e^{i2\pi\psi_{C_r}^{+im}},r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right):r=1,2,...,m\right\}$ be the collection of C_rCT-SF values and $E=(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r\in[0,1]$ and $\Sigma_{r=1}^m\alpha_r=1$.

Let
$$P^{-}=\left(\min_{1\le r\le m}\eta_{C_r}.e^{i2\pi\left[\min_{1\le r\le m}\eta_{C_r}^{+im}\right]},\max_{1\le r\le m}\psi_{C_r}^{-}.e^{i2\pi\left[\max_{1\le r\le m}\psi_{C_r}^{+im}\right]},\max_{1\le r\le m}\psi_{C_r}^{-}.e^{i2\pi\left[\max_{1\le r\le m}\psi_{C_r}^{+im}\right]},\max_{1\le r\le m}\eta_{C_r}.e^{i2\pi\left[\min_{1\le r\le m}\eta_{C_r}^{+im}\right]}\right)\text{ and}$$
$$P^{+}=\left(\max_{1\le r\le m}\eta_{C_r}.e^{i2\pi\left[\max_{1\le r\le m}\eta_{C_r}^{+im}\right]},\min_{1\le r\le m}\psi_{C_r}.e^{i2\pi\left[\min_{1\le r\le m}\psi_{C_r}^{+im}\right]},\min_{1\le r\le m}\psi_{C_r}.e^{i2\pi\left[\min_{1\le r\le m}\psi_{C_r}^{+im}\right]},\min_{1\le r\le m}\eta_{C_r}.e^{i2\pi\left[\max_{1\le r\le m}\eta_{C_r}^{+im}\right]}\right)\text{, where}$$
and $\lambda=1$. Then $P^{-}\le C_r^2CM$-PiFWGM$_E\left(P_1,P_2,P_3,...,P_m\right)\le P^{+}$.

Similar to the proof of Theorem 2.2.

Let $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}},\phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}},\psi_{C_r}e^{i2\pi\psi_{C_r}^{+im}},r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right):r=1,2,...,m\right\}$ and $\left\{P_r^*=\left(\eta_{C_r}^*e^{i2\pi\eta_{C_r}^{+im*}},\phi_{C_r}^*e^{i2\pi\phi_{C_r}^{+im*}},\psi_{C_r}^*e^{i2\pi\psi_{C_r}^{+im*}},r_{C_r}^*e^{i2\pi r_{C_r}^{+im*}}\right):r=1,2,...,m\right\}$ are two collections of C_rCT-SF values. If f $\eta_{C_r}\le\eta_{C_r}^*$, $\eta_{C_r}^{im}\le\eta_{C_r}^{im*}$, $\phi_{C_r}\ge\phi_{C_r}^*$, $\phi_{C_r}^{im}\ge\phi_{C_r}^{im*}$, $\psi_{C_r}\ge\psi_{C_r}^*$, $\psi_{C_r}^{im}\ge\psi_{C_r}^{im*}, r_{C_r}\le r_{C_r}^*$, and $r_{C_r}^{im}\le r_{C_r}^{im*}$, where $r=1,2,...,m$, then

C_rC-PiFWGM$_E\left(P_1,P_2,P_3,...,P_m\right)\le C_r^1C$-PiFWGM$_E\left(P_1^*,P_2^*,P_3^*,...,P_m^*\right)$.

Similar to the proof of Theorem 2.2.

Let $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}},\phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}},\psi_{C_r}e^{i2\pi\psi_{C_r}^{+im}},r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right):r=1,2,...,m\right\}$ and $\left\{P_r^*=\left(\eta_{C_r}^*e^{i2\pi\eta_{C_r}^{+im*}},\phi_{C_r}^*e^{i2\pi\phi_{C_r}^{+im*}},\psi_{C_r}^*e^{i2\pi\psi_{C_r}^{+im*}},r_{C_r}^*e^{i2\pi r_{C_r}^{+im*}}\right):r=1,2,...,m\right\}$ are two collections of C_rCT-SPF values. If f $\eta_{C_r}\le\eta_{C_r}^*$, $\eta_{C_r}^{im}\le\eta_{C_r}^{im*}$, $\phi_{C_r}\ge\phi_{C_r}^*$, $\phi_{C_r}^{im}\ge\phi_{C_r}^{im*}$, $\psi_{C_r}\ge\psi_{C_r}^*$, $\psi_{C_r}^{im}\ge\psi_{C_r}^{im*}, r_{C_r}\le r_{C_r}^*$, and $r_{C_r}^{im}\le r_{C_r}^{im*}$, where $r=1,2,...,m$, then

C_rC-PiFWGM$_E\left(P_1,P_2,P_3,...,P_m\right)\le C_r^2C$-PiFWGM$_E\left(P_1^*,P_2^*,P_3^*,...,P_m^*\right)$.

Similar to the proof of Theorem 2.2.

Let $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}},\phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}},\psi_{C_r}e^{i2\pi\psi_{C_r}^{+im}},r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right):r=1,2,...,m\right\}$ be the collection of C_rC-PiF values and let C_rC-PiFOWGM : $\Omega^m\to\Omega$, if
$$C_rC\text{-}PiFWGM_E\left(P_1,P_2,P_3,...,P_m\right)=\left(\left(P_{\delta(1)}^{\odot\alpha_1}\right)^{\lambda}\otimes\left(P_{\delta(2)}^{\odot\alpha_2}\right)^{\lambda}\otimes\left(P_{\delta(3)}^{\odot\alpha_3}\right)^{\lambda}\otimes...\otimes\left(P_{\delta(r)}^{\odot\alpha_m}\right)^{\lambda}\right)^{\frac{1}{\lambda}}$$
then C_rC-$PiFOWGM_E$ is called a cricular complex picture fuzzy ordered weighted geometric operator of dimension n, where $(\delta(1),\delta(2),...,\delta(m))$ is a permutation of $(1,2,...,m)$ such that $P_{\delta(r-1)}\ge P_{\delta(r)}$ for all r, Ω is the set of all C_rC-PiF values, $E=(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r\in[0,1]$ and $\Sigma_{r=1}^m\alpha_r=1$.

Let $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}},\phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}},\psi_{C_r}e^{i2\pi\psi_{C_r}^{+im}},r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right):r=1,2,...,m\right\}$ be the collection of C_rCT-SF values. Then by using the C_rC-PiFOWGM$_E$ operator their aggregated value is also a

C_rC-PiF value and

$$C_r^1C\text{-PiFOWGM}_E\left(P_1, P_2, P_3, ..., P_m\right)$$
$$=\begin{pmatrix} \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\eta_{C_{\delta(r)}}^{-}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\eta_{C_{\delta(r)}}^{-im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\psi_{C_{\delta(r)}}^{-}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right].e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\psi_{C_{\delta(r)}}^{-im}(u)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\psi_{C_{\delta(r)}}^{-}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right].e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\psi_{C_{\delta(r)}}^{-im}(u)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\eta_{C_{\delta(r)}}^{-}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\eta_{C_{\delta(r)}}^{-im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \end{pmatrix},$$

where $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_n)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\Sigma_{r=1}^{n}\alpha_r = 1$.

Similar to the proof of Theorem 2.2.

Let $\left\{P_r = \left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}}, \psi_{C_r}e^{i2\pi\psi_{C_r}^{+im}}, r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ be the collection of C_rC-PiF values. Let $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\Sigma_{r=1}^{m}\alpha_r = 1$. If $\left(\eta_{C_1}e^{i2\pi\eta_{C1}^{+im}}, \phi_{C_1}e^{i2\pi\phi_{C1}^{+im}}, \psi_{C_1}e^{i2\pi\psi_{C1}^{+im}}, r_{C_1}e^{i2\pi r_{C1}^{+im}}\right) =$
$= \left(\eta_{C_2}e^{i2\pi\eta_{C_2}^{+im}}, \phi_{C_2}e^{i2\pi\phi_{C_2}^{+im}}, \psi_{C_2}e^{i2\pi\psi_{C_2}^{+im}}, r_{C_2}e^{i2\pi r_{C_2}^{+im}}\right) = ...$
$= \left(\eta_{C_m}e^{i2\pi\eta_{Cm}^{+im}}, \phi_{C_m}e^{i2\pi\phi_{Cm}^{+im}}, \psi_{C_m}e^{i2\pi\psi_{Cm}^{+im}}, r_{C_m}e^{i2\pi r_{Cm}^{+im}}\right)$
$= \left(\eta_{C}e^{i2\pi\eta_{C}^{+im}}, \phi_{C}e^{i2\pi\phi_{C}^{+im}}, \psi_{C}e^{i2\pi\psi_{C}^{+im}}, r_{C}e^{i2\pi r_{C}^{+im}}\right)$ and $\lambda = 1$, then
C_r^1C-PiFOWGM$_E\left(P_1, P_2, P_3, ..., P_m\right) = \left(\eta_{C}e^{i2\pi\eta_{C}^{+im}}, \phi_{C}e^{i2\pi\phi_{C}^{+im}}, \psi_{C}e^{i2\pi\psi_{C}^{+im}}, r_{C}e^{i2\pi r_{C}^{+im}}\right)$.

Similar to the proof of Theorem 2.2.

Let $\left\{P_r = \left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}}, \psi_{C_r}e^{i2\pi\psi_{C_r}^{+im}}, r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ be the collection of C_rC-PiF values. Let $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\Sigma_{r=1}^{m}\alpha_r = 1$. If $\left(\eta_{C_1}e^{i2\pi\eta_{C1}^{+im}}, \phi_{C_1}e^{i2\pi\phi_{C1}^{+im}}, \psi_{C_1}e^{i2\pi\psi_{C1}^{+im}}, r_{C_1}e^{i2\pi r_{C1}^{+im}}\right) =$
$= \left(\eta_{C_2}e^{i2\pi\eta_{C_2}^{+im}}, \phi_{C_2}e^{i2\pi\phi_{C_2}^{+im}}, \psi_{C_2}e^{i2\pi\psi_{C_2}^{+im}}, r_{C_2}e^{i2\pi r_{C_2}^{+im}}\right) = ...$
$= \left(\eta_{C_m}e^{i2\pi\eta_{Cm}^{+im}}, \phi_{C_m}e^{i2\pi\phi_{Cm}^{+im}}, \psi_{C_m}e^{i2\pi\psi_{Cm}^{+im}}, r_{C_m}e^{i2\pi r_{Cm}^{+im}}\right)$
$= \left(\eta_{C}e^{i2\pi\eta_{C}^{+im}}, \phi_{C}e^{i2\pi\phi_{C}^{+im}}, \psi_{C}e^{i2\pi\psi_{C}^{+im}}, r_{C}e^{i2\pi r_{C}^{+im}}\right)$ and $\lambda = 1$, then
C_r^2C-PiFOWGM$_E\left(P_1, P_2, P_3, ..., P_m\right) = \left(\eta_{C}e^{i2\pi\eta_{C}^{+im}}, \phi_{C}e^{i2\pi\phi_{C}^{+im}}, \psi_{C}e^{i2\pi\psi_{C}^{+im}}, r_{C}e^{i2\pi r_{C}^{+im}}\right)$.

Similar to the proof of Theorem 2.2.

Let $\left\{P_r = \left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}}, \psi_{C_r}e^{i2\pi\psi_{C_r}^{+im}}, r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ be the collection of C_rC-PiF values and $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\Sigma_{r=1}^{m}\alpha_r = 1$, where $r = 1, 2, ..., m$.

Let
$P^- = \left(\min_{1\le r\le m}\eta_{C_r}.e^{i2\pi\left[\min_{1\le r\le m}\eta_{C_r}^{+im}\right]}, \max_{1\le r\le m}\psi_{C_r}^{-}.e^{i2\pi\left[\max_{1\le r\le m}\psi_{C_r}^{+im}\right]}, \max_{1\le r\le m}\psi_{C_r}^{-}.e^{i2\pi\left[\max_{1\le r\le m}\psi_{C_r}^{+im}\right]}, \min_{1\le r\le m}\eta_{C_r}.e^{i2\pi\left[\min_{1\le r\le m}\eta_{C_r}^{+im}\right]}\right)$ and

$$P^{+}=\left(\max_{1\le r\le m}\eta_{C_r}.e^{i2\pi\left[\max_{1\le r\le m}\eta_{C_r}^{+im}\right]},\min_{1\le r\le m}\psi_{C_r}.e^{i2\pi\left[\min_{1\le r\le m}\psi_{C_r}^{+im}\right]},\min_{1\le r\le m}\psi_{C_r}.e^{i2\pi\left[\min_{1\le r\le m}\psi_{C_r}^{+im}\right]},\max_{1\le r\le m}\eta_{C_r}.e^{i2\pi\left[\max_{1\le r\le m}\eta_{C_r}^{+im}\right]}\right)$$, where

and $\lambda=1$. Then $P^{-}\le C_r^1C-PiFOWGM_E\left(P_1,P_2,P_3,...,P_m\right)\le P^{+}$.

Similar to the proof of Theorem 2.2.

Let $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}},\phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}},\psi_{C_r}e^{i2\pi\psi_{C_r}^{+im}},r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right):r=1,2,...,m\right\}$ be the collection of C_rC-PiF values and $E=(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r\in[0,1]$ and $\Sigma_{r=1}^m\alpha_r=1$, where $r=1,2,...,m$.

Let

$$P^{-}=\left(\min_{1\le r\le m}\eta_{C_r}.e^{i2\pi\left[\min_{1\le r\le m}\eta_{C_r}^{+im}\right]},\max_{1\le r\le m}\psi_{C_r}^{-}.e^{i2\pi\left[\max_{1\le r\le m}\psi_{C_r}^{+im}\right]},\max_{1\le r\le m}\psi_{C_r}^{-}.e^{i2\pi\left[\max_{1\le r\le m}\psi_{C_r}^{+im}\right]},\max_{1\le r\le m}\eta_{C_r}.e^{i2\pi\left[\min_{1\le r\le m}\eta_{C_r}^{+im}\right]}\right)$$ and

$$P^{+}=\left(\max_{1\le r\le m}\eta_{C_r}.e^{i2\pi\left[\max_{1\le r\le m}\eta_{C_r}^{+im}\right]},\min_{1\le r\le m}\psi_{C_r}.e^{i2\pi\left[\min_{1\le r\le m}\psi_{C_r}^{+im}\right]},\min_{1\le r\le m}\psi_{C_r}.e^{i2\pi\left[\min_{1\le r\le m}\psi_{C_r}^{+im}\right]},\min_{1\le r\le m}\eta_{C_r}.e^{i2\pi\left[\max_{1\le r\le m}\eta_{C_r}^{+im}\right]}\right)$$, where

and $\lambda=1$. Then $P^{-}\le C_r^2C\text{-}PiFOWGM_E\left(P_1,P_2,P_3,...,P_m\right)\le P^{+}$.

Similar to the proof of Theorem 2.2.

Let Let $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}},\phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}},\psi_{C_r}e^{i2\pi\psi_{C_r}^{+im}},r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right):r=1,2,...,m\right\}$ and $\left\{P_r^*=\left(\eta_{C_r}^*e^{i2\pi\eta_{C_r}^{+im*}},\phi_{C_r}^*e^{i2\pi\phi_{C_r}^{+im*}},\psi_{C_r}^*e^{i2\pi\psi_{C_r}^{+im*}},r_{C_r}^*e^{i2\pi r_{C_r}^{+im*}}\right):r=1,2,...,m\right\}$ are two collections of C_rC-PiF values. If f $\eta_{C_r}\le\eta_{C_r}^*$, $\eta_{C_r}^{im}\le\eta_{C_r}^{im*}$, $\phi_{C_r}\ge\phi_{C_r}^*$, $\phi_{C_r}^{im}\ge\phi_{C_r}^{im*}$, $\psi_{C_r}\ge\psi_{C_r}^*$, $\psi_{C_r}^{im}\ge\psi_{C_r}^{im*}$, $r_{C_r}\le r_{C_r}^*$, and $r_{C_r}^{im}\le r_{C_r}^{im*}$ where $r=1,2,...,m$, then $C_r^1C\text{-}PiFOWGM_E\left(P_1,P_2,P_3,...,P_m\right)\le C_r^1C\text{-}PiFOWGM_E\left(P_1^*,P_2^*,P_3^*,...,P_m^*\right)$.

Similar to the proof of Theorem 2.2.

Let Let $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}},\phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}},\psi_{C_r}e^{i2\pi\psi_{C_r}^{+im}},r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right):r=1,2,...,m\right\}$ and $\left\{P_r^*=\left(\eta_{C_r}^*e^{i2\pi\eta_{C_r}^{+im*}},\phi_{C_r}^*e^{i2\pi\phi_{C_r}^{+im*}},\psi_{C_r}^*e^{i2\pi\psi_{C_r}^{+im*}},r_{C_r}^*e^{i2\pi r_{C_r}^{+im*}}\right):r=1,2,...,m\right\}$ are two collections of C_rC-PiF values. If f $\eta_{C_r}\le\eta_{C_r}^*$, $\eta_{C_r}^{im}\le\eta_{C_r}^{im*}$, $\phi_{C_r}\ge\phi_{C_r}^*$, $\phi_{C_r}^{im}\ge\phi_{C_r}^{im*}$, $\psi_{C_r}\ge\psi_{C_r}^*$, $\psi_{C_r}^{im}\ge\psi_{C_r}^{im*}$, $r_{C_r}\le r_{C_r}^*$, and $r_{C_r}^{im}\le r_{C_r}^{im*}$ where $r=1,2,...,m$, then $C_r^2C\text{-}PiFOWGM_E\left(P_1,P_2,P_3,...,P_m\right)\le C_r^2C\text{-}PiFOWGM_E\left(P_1^*,P_2^*,P_3^*,...,P_m^*\right)$.

Similar to the proof of Theorem 2.2.

4. MADM technique using CRITIC-WASPAS method based on circular complex picture fuzzy environment

Based on the evaluations of certain experts in the field of decision-making attributes, MADM (Multi-Attribute Decision Making) is considered a crucial method for selecting an option from a range of appealing alternatives. The decision-maker consistently aims to follow the most effective and rational course of action, making MADM particularly valuable in practical scenarios. Therefore, choosing the right decision-making approach is vital, and different techniques should be employed depending on the context. The WASPAS method, originally introduced by Hwang and Yoon [28] , focuses on proximity to the ideal solution. Additionally, Table 3 illustrates the rationale behind selecting the WASPAS method for ranking, offering a comparative overview of its features against various MADM techniques.

Table 3 Comparative assessment of Proposed WASPAS with traditional methods

Methods	Time	Simplicity	Mathematical evaluations	Stability	Information Types
AHP	High	Very Complex	Maximum	Poor	Mixed
VIKOR	Less	Simple	Moderate	Medium	Quantitative
ELECTRE	High	Moderately Complex	Moderate	Medium	Mixed
PROMETHEE	High	Moderately Complex	Moderate	Medium	Mixed
LINMAP	High	Moderately Complex	Maximum	Medium	Mixed
TOPSIS	Less	Simple	Minimum	Good	Quantitative
WASPAS	Very little	Very Simple	Minimum	Good	Quantitative

Unlike other weighting methods, the CRITIC technique incorporates statistical concepts. It assigns importance to criteria from a statistical stand point by utilizing measures such as the correlation coefficient and standard deviation. This characteristic allows CRITIC to produce distinctly different weight values for criteria an aspect often lacking in alternative weighting methods. The viewpoint adopted by the decision-maker (DM) becomes essential, and identifying this perspective in alignment with scientific principles is necessary before selecting a decision-making approach. A key benefit of the CRITIC method is its ability to normalize the decision matrix using the optimal values of the criteria simultaneously something not typically seen in other techniques. Therefore, in this section, we introduce a Circular Complex T-Spherical Fuzzy Set framework based on the proposed CRITIC-WASPAS model to address the MADM problem.

4.1 *CRITIC-WASPAS method*

Let $G = \{G_k : k = 1, 2, 3, ..., l\}$ be the set of experts, whose weight vector $w = (w_1, w_2, w_3, ..., w_l)$. Let $\mathcal{U} = \{C_i : i = 1, 2, 3, .., m\}$ be the set of alternatives and $P = \{P_r : r = 1, 2, 3, ..., n\}$ be the set of attributes. Let $Q^k = \left(\left(\eta^k_{it}, \phi^k_{it}, \psi^k_{it}, r^k_{it}\right)\right)_{n\times m}$ be the decision matrix (Information system) for each expert G_k. In this section, we construct a MADM methodology, including the following key steps:

Step 1 : Two categories of attributes, such as cost and benefits, are included in this procedure. Normalized the decision matrix to convert costs into benefits based on the following formula:

$$C^k_{ij} = \begin{cases} \left(\eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, \psi_{C_r} e^{i2\pi\psi^{+im}_{C_r}}, r_{C_r} e^{i2\pi r^{+im}_{C_r}}\right) & \text{for benefit attribute } P_r \\ \left(r_{C_r} e^{i2\pi r^{+im}_{C_r}}, \psi_{C_r} e^{i2\pi\psi^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, \eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}\right) & \text{for cost attribute } P_r \end{cases}$$

Step 2 : Compute the correlation coefficient Υ^k_{ij} between the attribute of each expert G_k, where

$$\Upsilon^k_{ij} = \frac{\sum\limits_{t=1}^{m}\left|\left(Sc\left(C_{it}\right)^{P_i} - Sc\left(C_i\right)^{P_i}\right)\right|\left|\left(Sc\left(C_{jt}\right)^{P_j} - Sc\left(C_j\right)^{P_j}\right)\right|}{\sqrt{\sum\limits_{t=1}^{m}\left(Sc\left(C_{it}\right)^{P_i} - Sc\left(C_i\right)^{P_i}\right)^2}\sqrt{\sum\limits_{t=1}^{m}\left(\left(Sc\left(C_{jt}\right)^{P_j} - Sc\left(C_j\right)^{P_j}\right)\right)^2}},$$

OR

$$\Upsilon^k_{ij} = \frac{\sum\limits_{t=1}^{m}\left|\left(H_{G_k}\left(C_{it}\right)^{P_i} - H_{G_k}\left(C_i\right)^{P_i}\right)\right|\left|\left(H_{G_k}\left(C_{jt}\right)^{P_j} - H_{G_k}\left(C_j\right)^{P_j}\right)\right|}{\sqrt{\sum\limits_{t=1}^{m}\left(H_{G_k}\left(C_{it}\right)^{P_i} - H_{G_k}\left(C_i\right)^{P_i}\right)^2}\sqrt{\sum\limits_{t=1}^{m}\left(\left(H_{G_k}\left(C_{jt}\right)^{P_j} - H_{G_k}\left(C_j\right)^{P_j}\right)\right)^2}},$$

$_{G_k}\left(C_i\right)^{P_i}$ and $_{G_k}\left(C_j\right)^{P_j}$ are the ith and jth $C_rCT\text{-}SFWAs$ or $C_rCT\text{-}SFWGs$ by applying Definition 2.2 or Definition 3, to aggregate the attribute P_i and P_j of each G_k, i.e.

$$\begin{aligned} &{}_{G_k}(C_i)^{P_i} \\ =\ & CrC\text{-}PiFWA_E\left({}_{G_k}(C_{i1})^{P_i}, {}_{G_k}(C_{i2})^{P_i}, {}_{G_k}(C_{i3})^{P_i}, ..., {}_{G_k}(C_{im})^{P_i}\right) \\ =\ & \begin{pmatrix} \left(\left(\sum_{r=1}^{m}\left(\alpha_j\left(\eta_{C_j}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(\sum_{j=1}^{m}\left(\alpha_r\left(\eta_{C_j}^{+im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1-\sum_{j=1}^{m}\left(\alpha_j\left(1-\left(\phi_{C_j}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(1-\sum_{j=1}^{m}\left(\alpha_j\left(1-\left(\phi_{C_j}^{+im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1-\sum_{j=1}^{m}\left(\alpha_j\left(1-\left(\psi_{C_j}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(1-\sum_{r=1}^{m}\left(\alpha_j\left(1-\left(\psi_{C_j}^{+im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(\sum_{j=1}^{m}\left(\alpha_j\left(r_{C_j}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(\sum_{j=1}^{m}\left(\alpha_j\left(r_{C_j}^{+im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)} \end{pmatrix}. \end{aligned}$$

or

$$\begin{aligned} {}_{G_k}(C_i)^{P_i} &= CrC\text{-}PiFWG_E\left({}_{G_k}(C_{i1})^{P_i}, {}_{G_k}(C_{i2})^{P_i}, {}_{G_k}(C_{i3})^{P_i}, ..., {}_{G_k}(C_{im})^{P_i}\right) \\ &= \begin{pmatrix} \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\eta_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\eta_{C_r}^{+im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\phi_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right].e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\phi_{C_r}^{+im}(u)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\psi_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right].e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\psi_{C_r}^{+im}(u)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(r_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(r_{C_r}^{+im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]} \end{pmatrix}, \end{aligned}$$

where $k = 1, 2, 3, ..., l$. Further $H_{G_k}(C_{it})^{P_i}$, $H_{G_k}(C_i)^{P_i}$, $H_{G_k}(C_{jt})^{P_j}$ and $H_{G_k}(C_j)^{P_j}$, are the accuracies of ${}_{G_k}(C_{it})^{P_i}$, ${}_{G_k}(C_i)^{P_i}$, ${}_{G_k}(C_{jt})^{P_j}$ and ${}_{G_k}(C_j)^{P_j}$. Furthermore, the weight of each ${}_{G_k}(C_{it})^{P_i}$ of each expert G_k is

$$\alpha_{it}^{k} = \frac{H_{G_k}(C_{it})^{P_i}}{\sum_{j=1}^{n} H\left(T_{jt}^{k}\right)}.$$

OR

$$\alpha_{it}^{k} = \frac{Sc_{G_k}(C_{it})^{P_i}}{\sum_{j=1}^{n} Sc\left(T_{jt}^{k}\right)}.$$

Step 3 : Compute the combined correlation coefficient Υ_{ij} is

$$\Upsilon_{ij} = \min\left\{\Upsilon_{ij}^{1}, \Upsilon_{ij}^{2}, ..., \Upsilon_{ij}^{l}\right\}.$$

Step 4 : Calculate the standard deviation $S_{SD_j^K}$ of each P_j for each expert G_k, where

$$SD_j^K = \sqrt{\frac{1}{m-1}\sum_{t=1}^{m}\left(H_{G_k}\left(C_{tj}\right)^{P_j} - H_{G_k}\left(C_j\right)^{P_j}\right)^2}.$$

OR

$$SD_j^K = \sqrt{\frac{1}{m-1}\sum_{t=1}^{m}\left(Sc_{G_k}\left(C_{tj}\right)^{P_j} - H_{G_k}\left(C_j\right)^{P_j}\right)^2}.$$

Step 5 : Calculate the simple standard deviation S_{SD_j} of each P_j, where

$$SD_j = \min\left\{SD_j^1, SD_j^2, SD_j^3, ..., SD_j^l\right\}.$$

Step 6 : Calculate the index $S_{Ind(P_j)}$ of each P_j by

$$Ind\left(P_j\right) = SD_j \sum_{t_1=1}^{n}\left(1 - \Upsilon_{t_1 j}\right).$$

Step 7 : Compute the weight of each P_j by

$$w_j = \frac{Ind\left(P_j\right)}{\sum\limits_{j=1}^{n} Ind\left(P_j\right)}.$$

Step 8 : Using C_rC-PiFWA and expert's weights, we can utilize overall ${}_{G_k}\left(C_{ij}\right)^{P_r}$ to ${}_{G}\left(C_{ij}\right)^{P_r}$ obtain the combine group decition matrix $G = \left[{}_{G}\left(C_{ij}\right)^{P_r}\right]_{m\times n}$, where ${}_{G}\left(C_{ij}\right)^{P_r} = \left(\eta_{C_r} e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r} e^{i2\pi\phi_{C_r}^{+im}}, \psi_{C_r} e^{i2\pi\psi_{C_r}^{+im}}, r_{C_r} e^{i2\pi r_{C_r}^{+im}}\right.$ and

$$\begin{aligned} {}_{G}\left(C_{ij}\right)^{P_r} &= C_rC\text{-}PiFWAM_E\left({}_{G_1}\left(C_{ij}\right)^{P_r}, {}_{G_2}\left(C_{ij}\right)^{P_r}, {}_{G_3}\left(C_{ij}\right)^{P_r}, ..., {}_{G_l}\left(C_{ij}\right)^{P_r}\right) \\ &= \begin{pmatrix} \left(\left(\sum\limits_{r=1}^{m}\left(\alpha_j\left(\eta_{C_j}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(\sum\limits_{j=1}^{m}\left(\alpha_r\left(\eta_{C_j}^{+im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1-\sum\limits_{j=1}^{m}\left(\alpha_j\left(1-\left(\phi_{C_j}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(1-\sum\limits_{j=1}^{m}\left(\alpha_j\left(1-\left(\phi_{C_j}^{+im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1-\sum\limits_{j=1}^{m}\left(\alpha_j\left(1-\left(\psi_{C_j}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(1-\sum\limits_{r=1}^{m}\left(\alpha_j\left(1-\left(\psi_{C_j}^{+im}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(\sum\limits_{j=1}^{m}\left(\alpha_j\left(r_{C_j}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(\sum\limits_{j=1}^{m}\left(\alpha_j\left(r_{C_j}^{+im}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)} \end{pmatrix}. \end{aligned}$$

Step 9 : Compute the weighted matrix ${}_wG = \left[{}_G\left(C_{ij}\right)^{w_r P_r}\right]_{m\times n} = \left[w_r \times_G \left(C_{ij}\right)^{P_r}\right]_{m\times n}$, where

$$w_r \times_G \left(C_{ij}\right)^{P_r} = \begin{pmatrix} \left(\left(\left(\alpha_j\left(\eta_{C_j}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(\left(\alpha_r\left(\eta^{+im}_{C_j}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1-\left(\alpha_j\left(1-\left(\phi_{C_j}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(1-\left(\alpha_j\left(1-\left(\phi^{+im}_{C_j}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(1-\left(\alpha_j\left(1-\left(\psi_{C_j}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(1-\left(\alpha_j\left(1-\left(\psi^{+im}_{C_j}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)}, \\ \left(\left(\left(\alpha_j\left(r_{C_j}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left(\left(\left(\alpha_j\left(r^{+im}_{C_j}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)} \end{pmatrix}.$$

and w_r is the weight vector of P_r which is obtained from Step 7.

Step 10 : Compute the Score of AM of weighted matrix ${}_wG$ using the following formula

$$Sc\left(AM\right) = Sc\left(P_1\right) = \frac{1}{8}\left\{\left(\eta_{C_1}\right)+\left(\eta^{im}_{C_1}\right)+\left(r_{C_1}\right)+\left(r^{im}_{C_1}\right)-\left(\psi_{C_1}\right)-\left(\psi^{im}_{C_1}\right)-\left(\phi_{C_1}\right)-\left(\phi^{im}_{C_1}\right)\right\}$$

OR

$$H\left(AM\right) = H\left(P_1\right) = \frac{1}{8}\left\{\left(\eta_{C_1}\right)+\left(\eta^{im}_{C_1}\right)+\left(\psi_{C_1}\right)+\left(\psi^{im}_{C_1}\right)+\left(\phi_{C_1}\right)+\left(\phi^{im}_{C_1}\right)+\left(r_{C_1}\right)+\left(r^{im}_{C_1}\right)\right\}.$$

Step 11 : Compute the GM of weighted matrix ${}_wG = \left[{}_G\left(C_{ij}\right)^{w_r P_r}\right]_{m\times n} = \left[w_r \times_G \left(C_{ij}\right)^{P_r}\right]_{m\times n}$, where

$$w_r \times_G \left(C_{ij}\right)^{P_r} = \begin{pmatrix} \left(\left(1-\left(\alpha_r\left(1-\left(\eta_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(1-\left(\alpha_r\left(1-\left(\eta^{+im}_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left[\left(\left(\left(\alpha_r\left(\phi_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right].e^{i2\pi\left[\left(\left(\left(\alpha_r\left(\phi^{+im}_{C_r}(u)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left[\left(\left(\left(\alpha_r\left(\psi_{C_r}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right].e^{i2\pi\left[\left(\left(\left(\alpha_r\left(\psi^{+im}_{C_r}(u)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]}, \\ \left(\left(1-\left(\alpha_r\left(1-\left(r_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).e^{i2\pi\left[\left(\left(1-\left(\alpha_r\left(1-\left(r^{+im}_{C_r}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right]} \end{pmatrix}$$

and w_r is the weight vector of P_r which is obtained from Step 7.
Step 12 : Compute the Score of GM of weighted matrix ${}_wG$ using the following formula

$$Sc\left(GM\right) = Sc\left(P_1\right) = \frac{1}{8}\left\{\left(\eta_{C_1}\right)+\left(\eta^{im}_{C_1}\right)+\left(r_{C_1}\right)+\left(r^{im}_{C_1}\right)-\left(\psi_{C_1}\right)-\left(\psi^{im}_{C_1}\right)-\left(\phi_{C_1}\right)-\left(\phi^{im}_{C_1}\right)\right\}.$$

OR

Compute the Accuracies of GM of weighted matrix ${}_wG$ using the following formula

$$H\left(GM\right)=H\left(P_1\right)=\frac{1}{8}\left\{\left(\eta_{C_1}\right)+\left(\eta_{C_1}^{im}\right)+\left(\psi_{C_1}\right)+\left(\psi_{C_1}^{im}\right)+\left(\phi_{C_1}\right)+\left(\phi_{C_1}^{im}\right)+\left(r_{C_1}\right)+\left(r_{C_1}^{im}\right)\right\}$$

Step 13 : Calculate WASPAS (Q_i) using the following formula

$$Q_i=\alpha\times Sc(AM)+(1-\alpha)\times Sc(GM)$$

OR

$$Q_i=\alpha\times H(AM)+(1-\alpha)\times H(GM)$$

Step 14 : Rank the preference order.

The ranking is determined by the Q_i values, higher Q_i values, higher the rank, and thus greater the alternative's results.

5. Analysis of stock market and financial sustainability

The stock market plays a vital role in the modern economic system, serving as a platform where investors, institutions, and corporations interact to allocate resources efficiently. It supports economic growth, innovation, and financial inclusion by facilitating the buying and selling of securities such as shares, bonds, and derivatives. The main goals of a sustainable stock market are to ensure transparency, investor confidence, risk management, and long-term financial stability. Through diversified investment strategies, regulatory frameworks, and advanced technological tools, the stock market enhances capital formation and economic resilience. In general, a well-functioning stock market contributes to sustainable economic growth, improved income distribution, and a stable financial environment. Various dimensions are considered in stock market development to guarantee investor welfare, economic viability, and market sustainability. These dimensions include:

(i) *Market Efficiency:*

Market efficiency refers to the degree to which stock prices reflect all available information. In an efficient market, securities are fairly valued, minimizing opportunities for arbitrage or speculative manipulation. The concept of market efficiency ensures that resources are allocated optimally, fostering investor trust and capital growth. Efficiency can be improved through transparency, rapid information dissemination, and effective regulatory mechanisms. It plays a central role in maintaining investor confidence, preventing bubbles, and supporting long-term economic development.

(ii) *Investment Efficiency:*

Investment efficiency in the stock market refers to the optimal allocation of financial resources across various sectors and securities to maximize returns while minimizing risks. Investors utilize data analytics, financial indicators, and multi-criteria decision-making models to evaluate investment opportunities. Efficient investment ensures that capital flows toward productive enterprises, leading to innovation, employment generation, and overall economic progress. Furthermore, technological tools such as algorithmic trading and artificial intelligence improve decision-making, reduce errors, and support sustainable portfolio management.

(iii) *Market Stability and Risk Management:*

Market stability is crucial for the smooth operation of financial systems and investor protection. Instability caused by economic shocks, political uncertainty, or speculative behavior can lead to market crashes and financial crises. Risk management strategies — including diversification, hedging, and regulatory supervision — help maintain equilibrium and prevent systemic failure. A stable market attracts long-term investors, promotes financial innovation, and ensures predictable economic growth. Sustainable market stability relies on strong governance, robust financial regulations, and global cooperation among institutions.

(iv) *Investor Welfare and Financial Well-being:*

Investor welfare represents the confidence, safety, and psychological comfort of market participants. A well-regulated stock market ensures investor protection, fair returns, and financial literacy. Transparency, corporate accountability, and ethical practices contribute to reducing fraud and market manipulation. Financial well-being extends beyond profit—it includes mental assurance, stable savings, and responsible investment habits. Modern markets integrate digital platforms, risk alerts, and investor education programs to enhance welfare and strengthen trust between the public and the financial system.

5.1 Importance of stock market and financial sustainability

The significance of the stock market in economic, social, and political dimensions is immense. It influences development, wealth distribution, employment, and technological innovation. Below is a detailed account of its importance:

(i) The stock market enables the mobilization of savings into productive investments, encouraging business expansion and economic development. It provides firms access to capital, allowing them to innovate, expand infrastructure, and increase employment. Furthermore, it supports entrepreneurship by providing a platform for public funding and venture investment.

(ii) Although short-term volatility is inherent, the stock market offers long-term financial growth opportunities. Investors benefit through dividends, capital gains, and portfolio diversification. Sustainable financial systems promote stability and protect investors from excessive risks, leading to a more secure economic future.

(iii) The stock market creates jobs directly and indirectly through financial institutions, brokerage firms, data analytics, and technology sectors. This expansion enhances national productivity, supports fintech development, and strengthens global financial competitiveness.

(iv) Through financial inclusion and education, the stock market improves societal welfare. It empowers individuals to participate in economic activities, reduces wealth inequality, and promotes long-term financial literacy. Additionally, ethical investment practices, such as ESG (Environmental, Social, and Governance) initiatives, align profit with social responsibility.

(v) The stock market also serves as an indicator of economic health. Fluctuations in market indices reflect macroeconomic conditions, investor sentiment, and government policies. Sustainable stock market practices contribute to national financial security and global economic stability.

(vi) Government regulations play a critical role in sustaining healthy stock markets. Policies encouraging transparency, corporate governance, and digital transformation protect investors and reduce fraudulent activities. International coordination among financial regulators ensures smooth market functioning across borders.

(vii) The rise of digital trading systems and AI-driven analytics is revolutionizing financial markets. These innovations enhance liquidity, accuracy, and real-time decision-making. Furthermore, sustainable financial strategies support climate-conscious investments, fintech innovation, and green financing—integrating finance with global sustainability goals.

In conclusion, the stock market is essential for achieving long-term financial sustainability, economic growth, and social development. It balances profitability with responsibility and provides the foundation for global economic progress. The main attributes of stock market analysis are as follows:

(1) *Growth Analysis:*

Growth analysis in the stock market examines the expansion of financial indices, investor participation, and sectoral performance over time. It helps identify economic trends, forecast returns, and guide policymakers in promoting stable economic growth. By understanding growth dynamics, investors and institutions can make informed and sustainable financial decisions.

(2) *Social Impact:*

The stock market's social impact extends beyond profits—it fosters economic inclusion, job creation, and public prosperity. By providing individuals access to investment opportunities, it reduces wealth inequality and strengthens financial independence. Moreover, ethical investing promotes socially responsible enterprises that contribute to environmental and social well-being.

(3) *Political Impact:*

Political stability directly affects stock market performance. Sound fiscal policies, anti-corruption measures, and investor-friendly legislation foster market confidence. Conversely, political unrest and policy uncertainty may trigger capital flight and reduce investor sentiment. A balanced relationship between politics and finance ensures long-term stability and growth.

(4) *Economic Impact:*

The economic impact of the stock market lies in its ability to stimulate investment, enhance innovation, and accelerate GDP growth. Stock exchanges channel capital into productive sectors such as technology, manufacturing, and services. They also promote competition, efficiency, and global integration of financial systems, which in turn drive sustainable economic prosperity.

(5) *Population Ratio:*

The relationship between population ratio and the stock market reflects how demographic trends influence investment behavior. Growing populations and rising middle-class segments increase the demand for investment products and financial services. Educating younger generations in financial literacy ensures a continuous flow of informed investors, sustaining long-term capital market development.

According to the growing demands of global investors, nations are focusing on digital transformation of their stock exchanges, improving transparency, accessibility, and sustainability to foster inclusive financial ecosystems.

Problem statement:

The stock market is a fundamental pillar of modern financial systems, enabling capital formation, investment diversification, and long-term economic sustainability. However, increasing market volatility, economic uncertainty, technological disruptions, and fluctuating investor behavior have made it challenging to evaluate the sustainability and performance of stock markets effectively. Decision-makers must thoroughly analyze financial indicators, market behavior, and socio-economic factors to select the most appropriate development strategy for achieving a sustainable and resilient stock market. Let $E = \{G_1\}$ be the set of experts with the weighting vector $(0.25,\ 0.45,\ 0.35)^T$. Let $\mathcal{U} = \{C_1, C_2, C_3, C_4\}$ be the set of alternatives representing key sustainability goals of the stock market, where C_1: Market Efficiency, C_2: Investment Efficiency, C_3: Market Stability and Risk Management, and C_4: Investor Welfare and Financial Well-being. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ be the set of evaluation attributes, where x_1: Growth Analysis, x_2: Social Impact, x_3: Political Impact, x_4: Economic Impact, and x_5: Population Ratio. Based on these attributes, our objective is to evaluate the best alternative among the four stock-market sustainability goals. These attributes were selected due to their strong relevance in financial market development, ensuring a comprehensive comparison of the sustainability dimensions. We mention that users may assign their preferred weight vector depending on policy priorities or expert recommendations. The information for the alternatives $x_i\ (i = 1, 2, 3, 4, 5)$ with respect to the criteria $C_t\ \{C_1, C_2, C_3, C_4\}$ is expressed using CrC-PiSFNs. Based on this information structure, we applied the CRITIC–WASPAS technique to determine the accuracy, weights, and ranking scores of each alternative. The corresponding information table provided by expert G_1 is displayed in Table 4, while Tables 5 and 6 present the computed attribute weights and decision scores according to the proposed method.

Table 4. Original decision matrix G_1

	x$_1$	x$_2$	x$_3$	x$_4$
c$_1$	$\begin{pmatrix} 0.11e^{i2\pi(0.11)}, \\ 0.02e^{i2\pi(0.02)}, \\ 0.09e^{i2\pi(0.09)}, \\ 0.21e^{i2\pi(021)} \end{pmatrix}$	$\begin{pmatrix} 0.21e^{i2\pi(0.21)}, \\ 0.03e^{i2\pi(0.03)}, \\ 0.18e^{i2\pi(0.18)}, \\ 0.31e^{i2\pi(0.31)} \end{pmatrix}$	$\begin{pmatrix} 0.31e^{i2\pi(0.31)}, \\ 0.04e^{i2\pi(0.04)}, \\ 0.27e^{i2\pi(0.27)}, \\ 0.11e^{i2\pi(0.11)} \end{pmatrix}$	$\begin{pmatrix} 0.41e^{i2\pi(0.41)}, \\ 0.05e^{i2\pi(0.05)}, \\ 0.36e^{i2\pi(0.36)}, \\ 0.44e^{i2\pi(0.44)} \end{pmatrix}$
c$_2$	$\begin{pmatrix} 0.12e^{i2\pi(0.12)}, \\ 0.03e^{i2\pi(0.03)}, \\ 0.09e^{i2\pi(0.09)}, \\ 0.44e^{i2\pi(0.44)} \end{pmatrix}$	$\begin{pmatrix} 0.22e^{i2\pi(0.22)}, \\ 0.04e^{i2\pi(0.04)}, \\ 0.18e^{i2\pi(0.18)}, \\ 0.33e^{i2\pi(0.33)} \end{pmatrix}$	$\begin{pmatrix} 0.32e^{i2\pi(0.32)}, \\ 0.05e^{i2\pi(0.05)}, \\ 0.27e^{i2\pi(0.27)}, \\ 0.22e^{i2\pi(0.22)} \end{pmatrix}$	$\begin{pmatrix} 0.42e^{i2\pi(0.42)}, \\ 0.06e^{i2\pi(0.06)}, \\ 0.36e^{i2\pi(0.36)}, \\ 0.55e^{i2\pi(0.55)} \end{pmatrix}$
c$_3$	$\begin{pmatrix} 0.13e^{i2\pi(0.13)}, \\ 0.04e^{i2\pi(0.04)}, \\ 0.09e^{i2\pi(0.09)}, \\ 0.22e^{i2\pi(0.22)} \end{pmatrix}$	$\begin{pmatrix} 0.23e^{i2\pi(0.23)}, \\ 0.05e^{i2\pi(0.05)}, \\ 0.18e^{i2\pi(0.18)}, \\ 0.33e^{i2\pi(0.33)} \end{pmatrix}$	$\begin{pmatrix} 0.33e^{i2\pi(0.33)}, \\ 0.06e^{i2\pi(0.06)}, \\ 0.27e^{i2\pi(0.27)}, \\ 0.22e^{i2\pi(0.22)} \end{pmatrix}$	$\begin{pmatrix} 0.34e^{i2\pi(0.34)}, \\ 0.07e^{i2\pi(0.07)}, \\ 0.36e^{i2\pi(0.36)} \\ 0.11e^{i2\pi(0.11)} \end{pmatrix}$
c$_4$	$\begin{pmatrix} 0.14e^{i2\pi(0.14)}, \\ 0.05e^{i2\pi(0.05)}, \\ 0.09e^{i2\pi(0.09)}, \\ 0.77e^{i2\pi(0.77)} \end{pmatrix}$	$\begin{pmatrix} 0.24e^{i2\pi(0.24)}, \\ 0.06e^{i2\pi(0.06)}, \\ 0.18e^{i2\pi(0.18)}, \\ 0.88e^{i2\pi(0.88)} \end{pmatrix}$	$\begin{pmatrix} 0.34e^{i2\pi(0.34)}, \\ 0.07e^{i2\pi(0.07)}, \\ 0.27e^{i2\pi(0.27)}, \\ 0.99e^{i2\pi(0.99)} \end{pmatrix}$	$\begin{pmatrix} 0.44e^{i2\pi(0.44)}, \\ 0.08e^{i2\pi(0.08)}, \\ 0.99e^{i2\pi(0.99)} \\ 0.66e^{i2\pi(0.66)} \end{pmatrix}$

According to CRITIC-WASPAS technique and based on Step 2, the score and weight of each alternative of expert G is given below in Tables 5 and 6, we have

Table 5. Represent the score of each alternative

	C_1	C_2	C_3	C_4
$SCO_G(x_1)$	0.05	0.11	0.055	0.19
$SCO_G(x_2)$	0.07	0.08	0.082	0.22
$SCO_G(x_3)$	0.02	0.05	0.055	0.24
$SCO_G(x_4)$	0.11	0.13	0.027	0.16

,

Table 6. Represent the weights of each alternative

	C_1	C_2	C_3	C_4
$_{G_1}w_{1i}$	0.1962	0.2857	0.25	0.23
$_{G_1}w_{2i}$	0.2897	0.2142	0.37	0.26
$_{G_1}w_{3i}$	0.1028	0.1428	0.25	0.30
$_{G_1}w_{4i}$	0.4112	0.3571	0.12	0.20

.

Further, the ith and jth C_rC-$PiFWA$s or C_rC-$PiFWG$s of $_G(C_i)$ and $_{G_k}(C_j)$ are given below by applying Definition 2.2 or Definition 3, to aggregate the attribute C_i and C_j of G. The results are shown in Tables 7

Table 7. Represent the C_rC-$PiFWA$ of $_G(C_i)$ based on C_i of G

	The C_rC-$PiFWA$ of $_G(C_i)$ based on C_i of G
$_G(C_1)$	$\begin{pmatrix} 0.2995e^{i2\pi(0.2995)}, \\ 0.03895e^{i2\pi(0.03895)}, \\ 0.2605e^{i2\pi(0.2605)}, \\ 0.3020e^{i2\pi(0.3020)} \end{pmatrix}$
$_G(C_2)$	$\begin{pmatrix} 0.3000e^{i2\pi(0.3000)}, \\ 0.9066e^{i2\pi(0.9066)}, \\ 0.2520e^{i2\pi(0.2520)}, \\ 0.9714e^{i2\pi(0.9714)} \end{pmatrix}$
$_G(C_3)$	$\begin{pmatrix} 0.3050e^{i2\pi(0.3050)}, \\ 0.0575e^{i2\pi(0.0575)}, \\ 0.2475e^{i2\pi(0.2475)}, \\ 0.2136e^{i2\pi(0.2136)} \end{pmatrix}$
$_G(C_4)$	$\begin{pmatrix} 0.30583e^{i2\pi(0.30583)}, \\ 0.0665e^{i2\pi(0.0665)}, \\ 0.2392e^{i2\pi(0.2392)}, \\ 0.8318e^{i2\pi(0.8318)} \end{pmatrix}$

Further, the Accuracy of C_rC-$PiFWA$ of $_G(C_i)$ based on C_i of G is given below in Table 8.

Table 8. Represent the the accuracy of C_rC-$PiFWAof_G(C_i)\,basedonC_iofG$

	C_1	C_2	C_3	C_4
$S_G(C_i)$	0.0755	0.02818	0.0534	0.2079

.

The correlation coefficient between each two attributes of expert G is determine by

$$\Upsilon_{ij}^{1} = \begin{pmatrix} 1 & 0.743594367 & 0.430019629 & 0.961968302 \\ 0.743594367 & 1 & 0.786622979 & 0.834335739 \\ 0.430019629 & 0.786622979 & 1 & 0.645279248 \\ 0.961968302 & 0.834335739 & 0.645279248 & 1 \end{pmatrix}.$$

Further, we calculate the standard deviation of each attribute C_i according to Step 4. The results are shown in Table 9.

Table 9. Represent the the standard deviation of each C_i

	C_1	C_2	C_3	C_4
	0.031730309	0.074684786	0.019510051	0.030793597

.

Determine the index value for every attribute in C_i. By using the attribute standard deviation and correlation coefficient, we can calculate the index of every attribute based on Step 6. The results are shown in Table 10.

Table 10. Represent the index of each C_i

	C_1	C_2	C_3	C_4
Index	0.027428241	0.047458217	0.022203963	0.017195659

.

From Table 8, we can calculate the weight of each C_i based on Step 7. Table 11 shows the weight of each attribute.

Table 11. Represent the weight of each C_i

	C_1	C_2	C_3	C_4
	0.239996338	0.415258072	0.194284055	0.150461536

.

Find the CrCT-SPFWA and CrCT-SPFWG on the basis weight from the table 9.The results are shown in Table 12.

Table 12. Represent $CrCT-$ SPFWA and $CrCT-$ SPFWG

	CrCT-SPFWA	CrCT-SPFWG
c_1	$\begin{pmatrix} 0.235521079e^{i2\pi(0.235521079)}, \\ 0.032552108e^{i2\pi(0.032552108)}, \\ 0.202968971e^{i2\pi(0.202968971)}, \\ 0.266703555e^{i2\pi(0.266703555)} \end{pmatrix}$	$\begin{pmatrix} 0.235521079e^{i2\pi(0.235521079)}, \\ 0.032552108e^{i2\pi(0.032552108)}, \\ 0.202968971e^{i2\pi(0.202968971)}, \\ 0.266703555e^{i2\pi(0.266703555)} \end{pmatrix}$
c_2	$\begin{pmatrix} 0.245521079e^{i2\pi(0.245521079)}, \\ 0.042552108e^{i2\pi(0.042552108)}, \\ 0.202968971e^{i2\pi(0.202968971)}, \\ 0.368129889e^{i2\pi(0.368129889)} \end{pmatrix}$	$\begin{pmatrix} 0.245521079e^{i2\pi(0.245521079)}, \\ 0.042552108e^{i2\pi(0.042552108)}, \\ 0.202968971e^{i2\pi(0.202968971)}, \\ 0.368129889e^{i2\pi(0.368129889)} \end{pmatrix}$
c_3	$\begin{pmatrix} 0.255521079e^{i2\pi(0.255521079)}, \\ 0.052552108e^{i2\pi(0.052552108)}, \\ 0.202968971e^{i2\pi(0.202968971)}, \\ 0.249127619e^{i2\pi(0.249127619)} \end{pmatrix}$	$\begin{pmatrix} 0.255521079e^{i2\pi(0.255521079)}, \\ 0.052552108e^{i2\pi(0.052552108)}, \\ 0.202968971e^{i2\pi(0.202968971)}, \\ 0.249127619e^{i2\pi(0.249127619)} \end{pmatrix}$
c_4	$\begin{pmatrix} 0.265521079e^{i2\pi(0.265521079)}, \\ 0.062552108e^{i2\pi(0.062552108)}, \\ 0.202968971e^{i2\pi(0.202968971)}, \\ 0.841870111e^{i2\pi(0.841870111)}, \end{pmatrix}$	$\begin{pmatrix} 0.265521079e^{i2\pi(0.265521079)}, \\ 0.062552108e^{i2\pi(0.062552108)}, \\ 0.202968971e^{i2\pi(0.202968971)}, \\ 0.841870111e^{i2\pi(0.841870111)}, \end{pmatrix}$

The score and weight of each alternative is given below in Tables 13

Table 13. Represent Score of AM and GM

	Score of CrCT-SFWA	Score of CrCT-SFWG
c_1	0.066675889	0.066675889
c_2	0.092032472	0.092032472
c_3	0.062281905	0.062281905
c_4	0.210467528	0.210467528

In order to have improved ranking accuracy and helpfulness of the decision-making process, in the CIRTIC WASPAS method, a more general equation for the total relative significance of alternatives. The results are shown in Table 14

Table 14. Represent WASPAS where $\alpha = 0.1$

$Alternatives$	α	$S(AM)$	$S(GM)$	$Q_1(C_1)$
C_1	0.1	0.066675889	0.066675889	0.066675889
C_2	0.1	0.092032472	0.092032472	0.092032472
C_3	0.1	0.062281905	0.062281905	0.062281905
C_4	0.1	0.210467528	0.210467528	0.210467528

$C_4 \succeq C_2 \succeq C_1 \succeq C_3$

Finally calculate the values Q_1,Q_2, Q_3, and Q_4. The results are shown in Table 15, 16, and 17.

Table 15. Score functions of the alternatives using different methods

Measure	$Q_1(C_1)$	$Q_2(C_2)$	$Q_3(C_3)$	$Q_4(C_4)$	Ranking
M. Akram et al. [29]	-	-	-	-	No Ranking
J. Qu et al. [30]	-	-	-	-	No Ranking
Z. Khan et al. [31]	-	-	-	-	No Ranking
P. Liu et al. [32]	-	-	-	-	No Ranking
Özen Özer [33]	-	-	-	-	No Ranking
CRITIC WASPAS $\lambda = 1, \beta = 0.1$	0.0667	0.0920	0.0623	0.2105	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$
$\lambda = 1, \beta = 0.2$	0.0667	0.0920	0.0623	0.2105	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$
$\lambda = 1, \beta = 0.3$	0.0667	0.0920	0.0623	0.2105	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$
$\lambda = 1, \beta = 0.4$	0.0667	0.0920	0.0623	0.2105	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$
$\lambda = 1, \beta = 0.5$	0.0667	0.0920	0.0623	0.2105	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$
$\lambda = 1, \beta = 0.6$	0.0667	0.0920	0.0623	0.2105	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$
$\lambda = 1, \beta = 0.7$	0.0667	0.0920	0.0623	0.2105	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$
$\lambda = 1, \beta = 0.8$	0.0667	0.0920	0.0623	0.2105	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$
$\lambda = 1, \beta = 0.9$	0.0667	0.0920	0.0623	0.2105	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$

Table 16. Ranking for $\lambda = 2$ and $\lambda = 3$ using Critic Waspas Method.

Measure	$Q_1(C_1)$	$Q_2(C_2)$	$Q_3(C_3)$	$Q_4(C_4)$	Ranking
$\lambda = 2, \beta = 0.1$	0.3457	0.3500	0.3447	0.3688	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$
$\lambda = 2, \beta = 0.2$	0.2664	0.2715	0.2656	0.2957	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$
$\lambda = 2, \beta = 0.3$	0.1871	0.1929	0.1865	0.2226	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$
$\lambda = 2, \beta = 0.4$	0.1079	0.1144	0.1074	0.1495	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$
$\lambda = 2, \beta = 0.5$	0.0286	0.0359	0.0284	0.0764	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$
$\lambda = 2, \beta = 0.6$	-0.0507	-0.0427	-0.0507	0.0033	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$
$\lambda = 2, \beta = 0.7$	-0.1299	-0.1212	-0.1298	-0.0699	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$
$\lambda = 2, \beta = 0.8$	-0.2092	-0.1998	-0.2089	-0.1430	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$
$\lambda = 2, \beta = 0.9$	-0.2885	-0.2783	-0.2880	-0.2161	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$
$\lambda = 3, \beta = 0.1$	0.3773	0.3779	0.3760	0.3842	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$
$\lambda = 3, \beta = 0.2$	0.2879	0.2892	0.2870	0.3015	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$
$\lambda = 3, \beta = 0.3$	0.1985	0.2006	0.1980	0.2189	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$
$\lambda = 3, \beta = 0.4$	0.1090	0.1119	0.1090	0.1362	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$
$\lambda = 3, \beta = 0.5$	0.0196	0.0232	0.0201	0.0536	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$
$\lambda = 3, \beta = 0.6$	-0.0699	-0.0654	-0.0689	-0.0291	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$
$\lambda = 3, \beta = 0.7$	-0.1593	-0.1541	-0.1579	-0.1118	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$
$\lambda = 3, \beta = 0.8$	-0.2487	-0.2427	-0.2469	-0.1944	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$
$\lambda = 3, \beta = 0.9$	-0.3382	-0.3314	-0.3359	-0.2771	$C_4 \succeq C_2 \succeq C_1 \succeq C_3$

Table 17. Ranking for $\lambda = 4$ and $\lambda = 5$ using Critic Waspas Method.

Measure	$Q_1(C_1)$	$Q_2(C_2)$	$Q_3(C_3)$	$Q_4(C_4)$	Ranking
$\lambda = 4, \beta = 0.1$	0.3830	0.3828	0.3816	0.3872	$C_4 \geq C_2 \geq C_1 \geq C_3$
$\lambda = 4, \beta = 0.2$	0.2916	0.2920	0.2907	0.3024	$C_4 \geq C_2 \geq C_1 \geq C_3$
$\lambda = 4, \beta = 0.3$	0.2001	0.2013	0.1998	0.2177	$C_4 \geq C_2 \geq C_1 \geq C_3$
$\lambda = 4, \beta = 0.4$	0.1087	0.1105	0.1089	0.1329	$C_4 \geq C_2 \geq C_1 \geq C_3$
$\lambda = 4, \beta = 0.5$	0.0172	0.0197	0.0180	0.0482	$C_4 \geq C_2 \geq C_1 \geq C_3$
$\lambda = 4, \beta = 0.6$	-0.0742	-0.0711	-0.0729	-0.0366	$C_4 \geq C_2 \geq C_1 \geq C_3$
$\lambda = 4, \beta = 0.7$	-0.1657	-0.1618	-0.1638	-0.1213	$C_4 \geq C_2 \geq C_1 \geq C_3$
$\lambda = 4, \beta = 0.8$	-0.2571	-0.2526	-0.2548	-0.2061	$C_4 \geq C_2 \geq C_1 \geq C_3$
$\lambda = 4, \beta = 0.9$	-0.3486	-0.3434	-0.3457	-0.2908	$C_4 \geq C_2 \geq C_1 \geq C_3$
$\lambda = 5, \beta = 0.1$	0.3845	0.3841	0.3831	0.3882	$C_4 \geq C_2 \geq C_1 \geq C_3$
$\lambda = 5, \beta = 0.2$	0.2925	0.2927	0.2916	0.3028	$C_4 \geq C_2 \geq C_1 \geq C_3$
$\lambda = 5, \beta = 0.3$	0.2005	0.2013	0.2002	0.2174	$C_4 \geq C_2 \geq C_1 \geq C_3$
$\lambda = 5, \beta = 0.4$	0.1085	0.1099	0.1088	0.1320	$C_4 \geq C_2 \geq C_1 \geq C_3$
$\lambda = 5, \beta = 0.5$	0.0165	0.0184	0.0174	0.0466	$C_4 \geq C_2 \geq C_1 \geq C_3$
$\lambda = 5, \beta = 0.6$	-0.0755	-0.0730	-0.0741	-0.0388	$C_4 \geq C_2 \geq C_1 \geq C_3$
$\lambda = 5, \beta = 0.7$	-0.1675	-0.1644	-0.1655	-0.1242	$C_4 \geq C_2 \geq C_1 \geq C_3$
$\lambda = 5, \beta = 0.8$	-0.2596	-0.2558	-0.2569	-0.2097	$C_4 \geq C_2 \geq C_1 \geq C_3$
$\lambda = 5, \beta = 0.9$	-0.3516	-0.3473	-0.3484	-0.2951	$C_4 \geq C_2 \geq C_1 \geq C_3$

Let $E = \{G_1, G_2, G_3\}$ be the set of experts with the weighting vector $(0.45, 0.35, 0.25)^T$. Let $\mathcal{U} = \{C_1, C_2, C_3, C_4\}$ be the set of alternatives representing key sustainability goals of the stock market, where C_1: Market Efficiency, C_2: Investment Efficiency, C_3: Market Stability and Risk Management, and C_4: Investor Welfare and Financial Well-being. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ be the set of evaluation attributes, where x_1: Growth Analysis, x_2: Social Impact, x_3: Political Impact, x_4: Economic Impact, and x_5: Population Ratio. Based on these attributes, our objective is to evaluate the best alternative among the four stock-market sustainability goals. These attributes were selected due to their strong relevance in financial market development, ensuring a comprehensive comparison of the sustainability dimensions. We mention that users may assign their preferred weight vector depending on policy priorities or expert recommendations. The information for the alternatives x_i $(i = 1, 2, 3, 4, 5)$ with respect to the criteria C_t $\{C_1, C_2, C_3, C_4\}$ is expressed using CrC-PiSFNs. Based on this information structure, we applied the CRITIC–WASPAS technique to determine the accuracy, weights, and ranking scores of each alternative. Thus, we considered our procedure and tried to evaluate our targeted result. Tables $18, 19$ and 20 represent the information table given by expert $G_1, G_2,$ and G_3.

Table 18. Original decision matrix G_1

	x_1	x_2	x_3	x_4
c_1	$\begin{pmatrix} 0.6e^{i2\pi(0.6)}, \\ 0.5e^{i2\pi(0.5)}, \\ 0.4e^{i2\pi(0.4)}, \\ 0.3e^{i2\pi(0.3)} \end{pmatrix}$	$\begin{pmatrix} 0.5e^{i2\pi(0.5)}, \\ 0.4e^{i2\pi(0.4)}, \\ 0.3e^{i2\pi(0.3)}, \\ 0.2e^{i2\pi(0.2)} \end{pmatrix}$	$\begin{pmatrix} 0.4e^{i2\pi(0.4)}, \\ 0.3e^{i2\pi(0.3)}, \\ 0.2e^{i2\pi(0.2)}, \\ 0.1e^{i2\pi(0.1)} \end{pmatrix}$	$\begin{pmatrix} 0.3e^{i2\pi(0.3)}, \\ 0.2e^{i2\pi(0.2)}, \\ 0.1e^{i2\pi(0.1)}, \\ 0.1e^{i2\pi(0.1)} \end{pmatrix}$
c_2	$\begin{pmatrix} 0.7e^{i2\pi(0.7)}, \\ 0.6e^{i2\pi(0.6)}, \\ 0.5e^{i2\pi(0.5)}, \\ 0.4e^{i2\pi(0.4)} \end{pmatrix}$	$\begin{pmatrix} 0.6e^{i2\pi(0.6)}, \\ 0.5e^{i2\pi(0.5)}, \\ 0.4e^{i2\pi(0.4)}, \\ 0.3e^{i2\pi(0.3)} \end{pmatrix}$	$\begin{pmatrix} 0.5e^{i2\pi(0.5)}, \\ 0.4e^{i2\pi(0.4)}, \\ 0.3e^{i2\pi(0.3)}, \\ 0.2e^{i2\pi(0.2)} \end{pmatrix}$	$\begin{pmatrix} 0.4e^{i2\pi(0.4)}, \\ 0.3e^{i2\pi(0.3)}, \\ 0.2e^{i2\pi(0.2)}, \\ 0.1e^{i2\pi(0.1)} \end{pmatrix}$
c_3	$\begin{pmatrix} 0.8e^{i2\pi(0.8)}, \\ 0.7e^{i2\pi(0.7)}, \\ 0.6e^{i2\pi(0.6)}, \\ 0.5e^{i2\pi(0.5)} \end{pmatrix}$	$\begin{pmatrix} 0.7e^{i2\pi(0.7)}, \\ 0.6e^{i2\pi(0.6)}, \\ 0.5e^{i2\pi(0.5)}, \\ 0.4e^{i2\pi(0.4)} \end{pmatrix}$	$\begin{pmatrix} 0.6e^{i2\pi(0.6)}, \\ 0.5e^{i2\pi(0.5)}, \\ 0.4e^{i2\pi(0.4)}, \\ 0.3e^{i2\pi(0.3)} \end{pmatrix}$	$\begin{pmatrix} 0.5e^{i2\pi(0.5)}, \\ 0.4e^{i2\pi(0.4)}, \\ 0.3e^{i2\pi(0.3)} \\ 0.2e^{i2\pi(0.2)} \end{pmatrix}$
c_4	$\begin{pmatrix} 0.9e^{i2\pi(0.9)}, \\ 0.8e^{i2\pi(0.8)}, \\ 0.7e^{i2\pi(0.7)}, \\ 0.6e^{i2\pi(0.6)} \end{pmatrix}$	$\begin{pmatrix} 0.8e^{i2\pi(0.8)}, \\ 0.7e^{i2\pi(0.7)}, \\ 0.6e^{i2\pi(0.6)}, \\ 0.5e^{i2\pi(0.5)} \end{pmatrix}$	$\begin{pmatrix} 0.7e^{i2\pi(0.7)}, \\ 0.6e^{i2\pi(0.6)}, \\ 0.5e^{i2\pi(0.5)}, \\ 0.4e^{i2\pi(0.4)} \end{pmatrix}$	$\begin{pmatrix} 0.6e^{i2\pi(0.6)}, \\ 0.5e^{i2\pi(0.5)}, \\ 0.4e^{i2\pi(0.4)} \\ 0.3e^{i2\pi(0.3)} \end{pmatrix}$

Table 19. Original decision matrix G_2

	x_1	x_2	x_3	x_4
c_1	$\begin{pmatrix} 0.5e^{i2\pi(0.5)}, \\ 0.6e^{i2\pi(0.6)}, \\ 0.7e^{i2\pi(0.7)}, \\ 0.8e^{i2\pi(0.8)} \end{pmatrix}$	$\begin{pmatrix} 0.4e^{i2\pi(0.4)}, \\ 0.5e^{i2\pi(0.5)}, \\ 0.6e^{i2\pi(0.6)}, \\ 0.7e^{i2\pi(0.7)} \end{pmatrix}$	$\begin{pmatrix} 0.3e^{i2\pi(0.3)}, \\ 0.4e^{i2\pi(0.4)}, \\ 0.5e^{i2\pi(0.5)}, \\ 0.6e^{i2\pi(0.6)} \end{pmatrix}$	$\begin{pmatrix} 0.2e^{i2\pi(0.2)}, \\ 0.3e^{i2\pi(0.3)}, \\ 0.4e^{i2\pi(0.4)}, \\ 0.5e^{i2\pi(0.5)} \end{pmatrix}$
c_2	$\begin{pmatrix} 0.4e^{i2\pi(0.4)}, \\ 0.5e^{i2\pi(0.5)}, \\ 0.6e^{i2\pi(0.6)}, \\ 0.7e^{i2\pi(0.7)} \end{pmatrix}$	$\begin{pmatrix} 0.3e^{i2\pi(0.3)}, \\ 0.4e^{i2\pi(0.4)}, \\ 0.5e^{i2\pi(0.5)}, \\ 0.6e^{i2\pi(0.6)} \end{pmatrix}$	$\begin{pmatrix} 0.2e^{i2\pi(0.2)}, \\ 0.3e^{i2\pi(0.3)}, \\ 0.4e^{i2\pi(0.4)}, \\ 0.5e^{i2\pi(0.5)} \end{pmatrix}$	$\begin{pmatrix} 0.1e^{i2\pi(0.1)}, \\ 0.2e^{i2\pi(0.2)}, \\ 0.3e^{i2\pi(0.3)}, \\ 0.4e^{i2\pi(0.4)} \end{pmatrix}$
c_3	$\begin{pmatrix} 0.3e^{i2\pi(0.3)}, \\ 0.4e^{i2\pi(0.4)}, \\ 0.5e^{i2\pi(0.5)}, \\ 0.6e^{i2\pi(0.6)} \end{pmatrix}$	$\begin{pmatrix} 0.2e^{i2\pi(0.2)}, \\ 0.3e^{i2\pi(0.3)}, \\ 0.4e^{i2\pi(0.4)}, \\ 0.5e^{i2\pi(0.5)} \end{pmatrix}$	$\begin{pmatrix} 0.1e^{i2\pi(0.1)}, \\ 0.2e^{i2\pi(0.2)}, \\ 0.3e^{i2\pi(0.3)}, \\ 0.4e^{i2\pi(0.4)} \end{pmatrix}$	$\begin{pmatrix} 0.1e^{i2\pi(0.1)}, \\ 0.1e^{i2\pi(0.1)}, \\ 0.2e^{i2\pi(0.2)} \\ 0.3e^{i2\pi(0.3)} \end{pmatrix}$
c_4	$\begin{pmatrix} 0.2e^{i2\pi(0.2)}, \\ 0.3e^{i2\pi(0.3)}, \\ 0.4e^{i2\pi(0.4)}, \\ 0.5e^{i2\pi(0.5)} \end{pmatrix}$	$\begin{pmatrix} 0.1e^{i2\pi(0.1)}, \\ 0.2e^{i2\pi(0.2)}, \\ 0.3e^{i2\pi(0.3)}, \\ 0.4e^{i2\pi(0.4)} \end{pmatrix}$	$\begin{pmatrix} 0.1e^{i2\pi(0.1)}, \\ 0.1e^{i2\pi(0.1)}, \\ 0.2e^{i2\pi(0.2)}, \\ 0.3e^{i2\pi(0.3)} \end{pmatrix}$	$\begin{pmatrix} 0.1e^{i2\pi(0.1)}, \\ 0.1e^{i2\pi(0.1)}, \\ 0.1e^{i2\pi(0.1)} \\ 0.2e^{i2\pi(0.2)} \end{pmatrix}$

Table 20. Original decision matrix G_3

	x$_1$	x$_2$	x$_3$	x$_4$
c$_1$	$\begin{pmatrix} 0.4e^{i2\pi(0.4)}, \\ 0.5e^{i2\pi(0.5)}, \\ 0.6e^{i2\pi(0.6)}, \\ 0.7e^{i2\pi(0.7)} \end{pmatrix}$	$\begin{pmatrix} 0.3e^{i2\pi(0.3)}, \\ 0.4e^{i2\pi(0.4)}, \\ 0.5e^{i2\pi(0.5)}, \\ 0.6e^{i2\pi(0.6)} \end{pmatrix}$	$\begin{pmatrix} 0.2e^{i2\pi(0.2)}, \\ 0.3e^{i2\pi(0.3)}, \\ 0.4e^{i2\pi(0.4)}, \\ 0.5e^{i2\pi(0.5)} \end{pmatrix}$	$\begin{pmatrix} 0.1e^{i2\pi(0.1)}, \\ 0.2e^{i2\pi(0.2)}, \\ 0.3e^{i2\pi(0.3)}, \\ 0.4e^{i2\pi(0.4)} \end{pmatrix}$
c$_2$	$\begin{pmatrix} 0.5e^{i2\pi(0.5)}, \\ 0.6e^{i2\pi(0.6)}, \\ 0.7e^{i2\pi(0.7)}, \\ 0.8e^{i2\pi(0.8)} \end{pmatrix}$	$\begin{pmatrix} 0.4e^{i2\pi(0.4)}, \\ 0.5e^{i2\pi(0.5)}, \\ 0.6e^{i2\pi(0.6)}, \\ 0.7e^{i2\pi(0.7)} \end{pmatrix}$	$\begin{pmatrix} 0.3e^{i2\pi(0.3)}, \\ 0.4e^{i2\pi(0.4)}, \\ 0.5e^{i2\pi(0.5)}, \\ 0.6e^{i2\pi(0.6)} \end{pmatrix}$	$\begin{pmatrix} 0.2e^{i2\pi(0.2)}, \\ 0.3e^{i2\pi(0.3)}, \\ 0.4e^{i2\pi(0.4)}, \\ 0.5e^{i2\pi(0.5)} \end{pmatrix}$
c$_3$	$\begin{pmatrix} 0.6e^{i2\pi(0.6)}, \\ 0.7e^{i2\pi(0.7)}, \\ 0.8e^{i2\pi(0.8)}, \\ 0.9e^{i2\pi(0.9)} \end{pmatrix}$	$\begin{pmatrix} 0.5e^{i2\pi(0.)}, \\ 0.6e^{i2\pi(0.)}, \\ 0.7e^{i2\pi(0.)}, \\ 0.8e^{i2\pi(0.)} \end{pmatrix}$	$\begin{pmatrix} 0.4e^{i2\pi(0.4)}, \\ 0.5e^{i2\pi(0.5)}, \\ 0.6e^{i2\pi(0.6)}, \\ 0.7e^{i2\pi(0.7)} \end{pmatrix}$	$\begin{pmatrix} 0.1e^{i2\pi(0.1)}, \\ 0.4e^{i2\pi(0.4)}, \\ 0.5e^{i2\pi(0.5)} \\ 0.6e^{i2\pi(0.6)} \end{pmatrix}$
c$_4$	$\begin{pmatrix} 0.7e^{i2\pi(0.7)}, \\ 0.7e^{i2\pi(0.7)}, \\ 0.8e^{i2\pi(0.8)}, \\ 0.9e^{i2\pi(0.9)} \end{pmatrix}$	$\begin{pmatrix} 0.6e^{i2\pi(0.)}, \\ 0.6e^{i2\pi(0.)}, \\ 0.7e^{i2\pi(0.)}, \\ 0.8e^{i2\pi(0.)} \end{pmatrix}$	$\begin{pmatrix} 0.5e^{i2\pi(0.5)}, \\ 0.5e^{i2\pi(0.5)}, \\ 0.6e^{i2\pi(0.6)}, \\ 0.7e^{i2\pi(0.7)} \end{pmatrix}$	$\begin{pmatrix} 0.4e^{i2\pi(0.4)}, \\ 0.4e^{i2\pi(0.4)}, \\ 0.5e^{i2\pi(0.5)} \\ 0.6e^{i2\pi(0.6)} \end{pmatrix}$

According to CRITIC-WASPAS technique and based on Step 2, the accuracy and weight of each alternative of expert G_1, the accuracy and weight of each alternative of expert G_2, and the accuracy and weight of each alternative of expert G_3.

Further, the ith and jth C_rC-$PiFWAs$ or C_rC-$PiFWGs$ of $_{G_k}(C_i)^{P_i}$ and $_{G_k}(C_j)^{P_j}$ are given below by applying Definition 2.2 or Definition 3, to aggregate the

Table 21. Represent the C_rC-$PiFWA$ of $_{G_1}(C_i)^{P_i}$ based on C_i of G_1, G_2, and G_3

	C_rC-$PiFWA$ of G_1	C_rC-$PiFWA$ of G_2	C_rC-$PiFWA$ of G_3
$_{G_1}(C_1)^{P_1}$	$\begin{pmatrix} 0.4877e^{i2\pi(0.4877)}, \\ 0.3877e^{i2\pi(0.3877)}, \\ 0.2877e^{i2\pi(0.2877)}, \\ 0.2020e^{i2\pi(0.2020)} \end{pmatrix}$	$\begin{pmatrix} 0.3750e^{i2\pi(0.3750)}, \\ 0.4750e^{i2\pi(0.4750)}, \\ 0.5750e^{i2\pi(0.5750)}, \\ 0.6750e^{i2\pi(0.6750)} \end{pmatrix}$	$\begin{pmatrix} 0.2812e^{i2\pi(0.2812)}, \\ 0.3812e^{i2\pi(0.3812)}, \\ 0.4812e^{i2\pi(0.4812)}, \\ 0.5812e^{i2\pi(0.5812)} \end{pmatrix}$
$_{G_1}(C_2)^{P_2}$	$\begin{pmatrix} 0.5812e^{i2\pi(0.5812)}, \\ 0.4812e^{i2\pi(0.4812)}, \\ 0.3812e^{i2\pi(0.3812)}, \\ 0.2812e^{i2\pi(0.2812)} \end{pmatrix}$	$\begin{pmatrix} 0.2812e^{i2\pi(0.2812)}, \\ 0.3812e^{i2\pi(0.3812)}, \\ 0.4812e^{i2\pi(0.4812)}, \\ 0.5812e^{i2\pi(0.5812)} \end{pmatrix}$	$\begin{pmatrix} 0.3750e^{i2\pi(0.3750)}, \\ 0.4750e^{i2\pi(0.4750)}, \\ 0.5750e^{i2\pi(0.5750)}, \\ 0.6750e^{i2\pi(0.6750)} \end{pmatrix}$
$_{G_1}(C_3)^{P_3}$	$\begin{pmatrix} 0.6750e^{i2\pi(0.6750)}, \\ 0.5750e^{i2\pi(0.5750)}, \\ 0.4750e^{i2\pi(0.4750)}, \\ 0.3750e^{i2\pi(0.3750)} \end{pmatrix}$	$\begin{pmatrix} 0.2020e^{i2\pi(0.2020)}, \\ 0.2877e^{i2\pi(0.2877)}, \\ 0.3877e^{i2\pi(0.3877)}, \\ 0.4877e^{i2\pi(0.4877)} \end{pmatrix}$	$\begin{pmatrix} 0.4404e^{i2\pi(0.4404)}, \\ 0.5744e^{i2\pi(0.5744)}, \\ 0.6744e^{i2\pi(0.6744)}, \\ 0.7744e^{i2\pi(0.7744)} \end{pmatrix}$
$_{G_1}(C_4)^{P_4}$	$\begin{pmatrix} 0.7708e^{i2\pi(0.7708)}, \\ 0.6708e^{i2\pi(0.6708)}, \\ 0.5708e^{i2\pi(0.5708)}, \\ 0.4708e^{i2\pi(0.4708)} \end{pmatrix}$	$\begin{pmatrix} 0.1388e^{i2\pi(0.1388)}, \\ 0.2055e^{i2\pi(0.2055)}, \\ 0.2916e^{i2\pi(0.2916)}, \\ 0.3916e^{i2\pi(0.3916)} \end{pmatrix}$	$\begin{pmatrix} 0.5700e^{i2\pi(0.5700)}, \\ 0.5700e^{i2\pi(0.5700)}, \\ 0.6700e^{i2\pi(0.6700)}, \\ 0.7700e^{i2\pi(0.7700)} \end{pmatrix}$

Based on Step 3, the combined correlation coefficient Υ_{ij} is

$$\Upsilon_{ij} = \begin{pmatrix} 1 & 0.996561717 & 0.992331783. & 0.987886091 \\ 0.996561717 & 1 & 0.994634873 & 0.992489122 \\ 0.992331783 & 0.994634873 & 1 & 0.993809195 \\ 0.987886091 & 0.992489122 & 0.993809195 & 1 \end{pmatrix}.$$

Using Step 5, we calculate the simple standard deviation SD_j of each C_j. The results are shown in Table 22.

Table 22. Represent the the standard deviation of each P_i

	C_1	C_2	C_3	C_4
SD_i	0.109419151	0.114564392	0.109419151	0.090597724

Determine the index value for every attribute in C_i. By using the attribute standard deviation and correlation coefficient, we can calculate the index of every attribute based on Step 6. The results are shown in Table 23.

Table 23. Represent the index of each C_i

	C_1	C_2	C_3	C_4
Index	0.002540757	0.001869036	0.00210349	0.002338834

From Table 23, we can calculate the weight of each C_i based on Step 7. Table 24 shows the weight of each attribute.

Table 24. Represent the weight of each C_i

	C_1	C_2	C_3	C_4
	0.287022548	0.211140037	0.237625623	0.264211792

A more effective technique is needed to select the optimal scheme among several when the attribute weight is unidentified, a circumstance that CRITIC can manage. The following step provides the optimal ranking and selection criteria. Based on Step 8,using C_rC-$PiFWA$ and expert's weights, we can utilize overall $_{G_k}\left(C_{ij}\right)^{P_r}$to $_G\left(C_{ij}\right)^{P_r}$ obtain the combine group decition matrix $G = \left[_G\left(C_{ij}\right)^{P_r}\right]_{m\times n}$, where $_G\left(C_{ij}\right)^{P_r} = \left\langle \eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, \psi_{C_r} e^{i2\pi\psi^{+im}_{C_r}}, r_{C_r} e^{i2\pi r^{+im}_{C_r}} \right\rangle$, where $i = 1, 2, 3, ...m, j = 1, 2, 3, ...n$. Table 24 shows the matrix $G = \left[_G\left(C_{ij}\right)^{P_r}\right]_{m\times n}$,where $w_{G_1} = 0.25$, $w_{G_2} = 0.45$ and $w_{G_3} = 0.2$.

Table 25 Represent the combine group decision matrix G

	x$_1$	x$_2$	x$_3$	x$_4$
c$_1$	$\begin{pmatrix} 0.545e^{i2\pi(0.545)}, \\ 0.51e^{i2\pi(0.51)}, \\ 0.475e^{i2\pi(0.475)}, \\ 0.59e^{i2\pi(0.59)} \end{pmatrix}$	$\begin{pmatrix} 0.44e^{i2\pi(0.44)}, \\ 0.405e^{i2\pi(0.405)}, \\ 0.58e^{i2\pi(0.58)}, \\ 0.485e^{i2\pi(0.485)} \end{pmatrix}$	$\begin{pmatrix} 0.318e^{i2\pi(0.318)}, \\ 0.548e^{i2\pi(0.548)}, \\ 0.452e^{i2\pi(0.452)}, \\ 0.448e^{i2\pi(0.448)} \end{pmatrix}$	$\begin{pmatrix} 0.23e^{i2\pi(0.23)}, \\ 0.195e^{i2\pi(0.195)}, \\ 0.79e^{i2\pi(0.79)}, \\ 0.32e^{i2\pi(0.32)} \end{pmatrix}$
c$_2$	$\begin{pmatrix} 0.58e^{i2\pi(0.58)}, \\ 0.545e^{i2\pi(0.545)}, \\ 0.44e^{i2\pi(0.44)}, \\ 0.625e^{i2\pi(0.625)} \end{pmatrix}$	$\begin{pmatrix} 0.475e^{i2\pi(0.475)}, \\ 0.44e^{i2\pi(0.44)}, \\ 0.545e^{i2\pi(0.545)}, \\ 0.52e^{i2\pi(0.52)} \end{pmatrix}$	$\begin{pmatrix} 0.37e^{i2\pi(0.37)}, \\ 0.335e^{i2\pi(0.335)}, \\ 0.65e^{i2\pi(0.65)}, \\ 0.415e^{i2\pi(0.415)} \end{pmatrix}$	$\begin{pmatrix} 0.265e^{i2\pi(0.265)}, \\ 0.23e^{i2\pi(0.23)}, \\ 0.755e^{i2\pi(0.755)}, \\ 0.31e^{i2\pi(0.31)} \end{pmatrix}$
c$_3$	$\begin{pmatrix} 0.615e^{i2\pi(0.615)}, \\ 0.58e^{i2\pi(0.0.58)}, \\ 0.405e^{i2\pi(0.405)}, \\ 0.66e^{i2\pi(0.66)} \end{pmatrix}$	$\begin{pmatrix} 0.51e^{i2\pi(0.51)}, \\ 0.475e^{i2\pi(0.475)}, \\ 0.51e^{i2\pi(0.51)}, \\ 0.555e^{i2\pi(0.555)} \end{pmatrix}$	$\begin{pmatrix} 0.405e^{i2\pi(0.405)}, \\ 0.37e^{i2\pi(0.37)}, \\ 0.615e^{i2\pi(0.615)}, \\ 0.45e^{i2\pi(0.45)} \end{pmatrix}$	$\begin{pmatrix} 0.285e^{i2\pi(0.285)}, \\ 0.265e^{i2\pi(0.265)}, \\ 0.72e^{i2\pi(0.72)}, \\ 0.345e^{i2\pi(0.345)} \end{pmatrix}$
c$_4$	$\begin{pmatrix} 0.65e^{i2\pi(0.65)}, \\ 0.59e^{i2\pi(0.59)}, \\ 0.395e^{i2\pi(0.395)}, \\ 0.67e^{i2\pi(0.67)} \end{pmatrix}$	$\begin{pmatrix} 0.545e^{i2\pi(0.545)}, \\ 0.485e^{i2\pi(0.485)}, \\ 0.5e^{i2\pi(0.5)}, \\ 0.565e^{i2\pi(0.565)} \end{pmatrix}$	$\begin{pmatrix} 0.475e^{i2\pi(0.475)}, \\ 0.38e^{i2\pi(0.38)}, \\ 0.605e^{i2\pi(0.605)}, \\ 0.46e^{i2\pi(0.46)} \end{pmatrix}$	$\begin{pmatrix} 0.405e^{i2\pi(0.405)}, \\ 0.31e^{i2\pi(0.31)}, \\ 0.71e^{i2\pi(0.71)}, \\ 0.355e^{i2\pi(0.355)} \end{pmatrix}$

Using the data of table 25 to calculate the final values Q_1,Q_2, Q_3, and Q_4. The results are shown in Table 26,.27, and 28

Table 26. Score functions of the alternatives using different methods

Measure	$Q_1(C_1)$	$Q_2(C_2)$	$Q_3(C_3)$	$Q_4(C_4)$	Ranking
Kadyan *et al.* [34]	-	-	-	-	No Ranking
Mahmood *et al.* [35]	-	-	-	-	No Ranking
Garg [36]	-	-	-	-	No Ranking
Wei [37]	-	-	-	-	No Ranking
Wei [38]	-	-	-	-	No Ranking
Ali *et al.* [20]	-	-	-	-	No Ranking
CRITIC-WASPAS $\lambda = 1, \beta = 0.1$	0.4593	0.4699	0.4864	0.5076	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$
$\lambda = 1, \beta = 0.2$	0.4593	0.4699	0.4864	0.5076	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$
$\lambda = 1, \beta = 0.3$	0.4593	0.4699	0.4864	0.5076	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$
$\lambda = 1, \beta = 0.4$	0.4593	0.4699	0.4864	0.5076	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$
$\lambda = 1, \beta = 0.5$	0.4593	0.4699	0.4864	0.5076	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$
$\lambda = 1, \beta = 0.6$	0.4593	0.4699	0.4864	0.5076	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$
$\lambda = 1, \beta = 0.7$	0.4593	0.4699	0.4864	0.5076	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$
$\lambda = 1, \beta = 0.8$	0.4593	0.4699	0.4864	0.5076	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$
$\lambda = 1, \beta = 0.9$	0.4593	0.4699	0.4864	0.5076	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$

Table 27. Ranking for $\lambda = 2$ and $\lambda = 3$ using Critic Waspas Method.

Measure	$Q_1(C_1)$	$Q_2(C_2)$	$Q_3(C_3)$	$Q_4(C_4)$	Ranking
$\lambda = 2, \beta = 0.1$	0.6058	0.6067	0.6095	0.6136	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$
$\lambda = 2, \beta = 0.2$	0.6044	0.6059	0.6093	0.6139	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$
$\lambda = 2, \beta = 0.3$	0.6030	0.6051	0.6092	0.6143	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$
$\lambda = 2, \beta = 0.4$	0.6016	0.6042	0.6090	0.6147	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$
$\lambda = 2, \beta = 0.5$	0.6003	0.6034	0.6088	0.6150	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$
$\lambda = 2, \beta = 0.6$	0.5989	0.6026	0.6086	0.6154	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$
$\lambda = 2, \beta = 0.7$	0.5975	0.6018	0.6085	0.6158	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$
$\lambda = 2, \beta = 0.8$	0.5961	0.6010	0.6083	0.6161	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$
$\lambda = 2, \beta = 0.9$	0.5947	0.6002	0.6081	0.6165	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$
$\lambda = 3, \beta = 0.1$	0.6033	0.6028	0.6037	0.6049	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$
$\lambda = 3, \beta = 0.2$	0.6019	0.6021	0.6036	0.6054	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$
$\lambda = 3, \beta = 0.3$	0.6006	0.6014	0.6035	0.6058	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$
$\lambda = 3, \beta = 0.4$	0.5992	0.6007	0.6034	0.6063	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$
$\lambda = 3, \beta = 0.5$	0.5979	0.6000	0.6033	0.6067	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$
$\lambda = 3, \beta = 0.6$	0.5965	0.5992	0.6033	0.6071	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$
$\lambda = 3, \beta = 0.7$	0.5952	0.5985	0.6032	0.6076	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$
$\lambda = 3, \beta = 0.8$	0.5938	0.5978	0.6031	0.6080	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$
$\lambda = 3, \beta = 0.9$	0.5925	0.5971	0.6030	0.6085	$C_4 \succeq C_3 \succeq C_2 \succeq C_1$

Table 28. Ranking for $\lambda = 4$ and $\lambda = 5$ using Critic Waspas Method.

Measure	$Q_1(C_1)$	$Q_2(C_2)$	$Q_3(C_3)$	$Q_4(C_4)$	Ranking
$\lambda = 4, \beta = 0.1$	0.5971	0.5966	0.5971	0.5978	$C_4 \geq C_3 \geq C_2 \geq C_1$
$\lambda = 4, \beta = 0.2$	0.5959	0.5960	0.5971	0.5983	$C_4 \geq C_3 \geq C_2 \geq C_1$
$\lambda = 4, \beta = 0.3$	0.5947	0.5954	0.5971	0.5987	$C_4 \geq C_3 \geq C_2 \geq C_1$
$\lambda = 4, \beta = 0.4$	0.5935	0.5948	0.5971	0.5992	$C_4 \geq C_3 \geq C_2 \geq C_1$
$\lambda = 4, \beta = 0.5$	0.5923	0.5943	0.5971	0.5996	$C_4 \geq C_3 \geq C_2 \geq C_1$
$\lambda = 4, \beta = 0.6$	0.5911	0.5937	0.5971	0.6001	$C_4 \geq C_3 \geq C_2 \geq C_1$
$\lambda = 4, \beta = 0.7$	0.5899	0.5931	0.5971	0.6005	$C_4 \geq C_3 \geq C_2 \geq C_1$
$\lambda = 4, \beta = 0.8$	0.5888	0.5925	0.5971	0.6010	$C_4 \geq C_3 \geq C_2 \geq C_1$
$\lambda = 4, \beta = 0.9$	0.5876	0.5919	0.5971	0.6014	$C_4 \geq C_3 \geq C_2 \geq C_1$
$\lambda = 5, \beta = 0.1$	0.5936	0.5931	0.5937	0.5942	$C_4 \geq C_3 \geq C_2 \geq C_1$
$\lambda = 5, \beta = 0.2$	0.5925	0.5927	0.5937	0.5946	$C_4 \geq C_3 \geq C_2 \geq C_1$
$\lambda = 5, \beta = 0.3$	0.5914	0.5922	0.5938	0.5951	$C_4 \geq C_3 \geq C_2 \geq C_1$
$\lambda = 5, \beta = 0.4$	0.5903	0.5917	0.5938	0.5955	$C_4 \geq C_3 \geq C_2 \geq C_1$
$\lambda = 5, \beta = 0.5$	0.5893	0.5912	0.5939	0.5960	$C_4 \geq C_3 \geq C_2 \geq C_1$
$\lambda = 5, \beta = 0.6$	0.5882	0.5907	0.5940	0.5965	$C_4 \geq C_3 \geq C_2 \geq C_1$
$\lambda = 5, \beta = 0.7$	0.5871	0.5903	0.5940	0.5969	$C_4 \geq C_3 \geq C_2 \geq C_1$
$\lambda = 5, \beta = 0.8$	0.5860	0.5898	0.5941	0.5974	$C_4 \geq C_3 \geq C_2 \geq C_1$
$\lambda = 5, \beta = 0.9$	0.5850	0.5893	0.5942	0.5978	$C_4 \geq C_3 \geq C_2 \geq C_1$

6. Comparative analysis of the proposed methods

This section aims to highlight the advantages and effectiveness of the proposed decision-making methodology through quantitative comparison with several existing approaches. The detailed comparison procedure is outlined below.

6.1 Comparative analysis for picture fuzzy set environment

Based on the data in Tables 4, 14, 15, 16, and 17, we compare the suggested techniques (CRITIC-WASPAS, CrC-$PiFWAM$, and CrC-$PiFWGM$) with several existing operators. When comparing the suggested techniques with the existing approaches, the best and top-ranked alternatives remain the same, demonstrating the reliability and practicability of the proposed work. The existing operators employed for comparison in this study include: Akram *et al.* [29], who proposed a decision-making model under complex picture fuzzy Hamacher aggregation operators; Qu *et al.* [30], who developed an innovative decision-making approach based on correlation coefficients of complex picture fuzzy sets with applications in cluster analysis; Khan *et al.* [31], who introduced distance measures and their applications to decision making, medical diagnosis, and pattern recognition under complex picture fuzzy sets; Liu *et al.* [32], who extended power aggregation operators for decision making based on complex picture fuzzy knowledge; and Özer [33], who proposed Hamacher prioritized aggregation operators based on complex picture fuzzy sets and their applications in decision-making problems.

In comparison to other approaches already in use, our suggested alternative selection methodology based on the CrC-$PiFWAM$ and CrC-$PiFWGM$ techniques with entropy measure is more flexible and efficient. According to all procedures, C_4 is the best alternative, as indicated in Tables 15, 16 and 17, however various computation stages are used by various techniques. For instance, the authors of the current approaches employed the Crcular Complex Picture fuzzy weighted arithmetic mean and Crcular Complex Picture fuzzy weighted geometric mean operators to aggregate the information, however in the suggested method, we used the CrC-$PiFWAM$ and CrC-$PiFWGM$ methodologies with weight to aggregate crcular complex Picture fuzzy information. Furthermore, we employed various λ, β values and ranked the $P_i(i = 1, 2, 3, 4)$ options according to the CrC-$PiFWAM$ and CrC-$PiFWGM$ operator in Tables 15, 16 and 17.

In addition, we observed some intriguing discoveries, which are explained in more detail below:

1. Akram *et al.* [29], Qu *et al.* [30], Khan *et al.* [31], Liu *et al.* [32], and Özer [33] developed operators that facilitate decision-making regarding parameter values and provide effective measures. However, it should be noted that these operators only handle membership, non-membership, and hesitant degrees in the complex form.

   numbers, leading to frequent destruction of information. It follows that the suggested work is better than these existing approaches because we also define the radius .

2. The ideal option does not change, according to the results displayed in Tables 15 16 and 17, although the theories structures are completely different. For example, in the context of the PiSFS, we calculated the results by setting the C-PiFS judgment of the CrC-PiFS to zero, and setting the Crq-ROFS judgment to zero for the CT-SFSs. Consequently, some information will be lost in the findings that relate to them. As indicated in Tables 15, 16 and 17, this causes the findings of the suggested technique to be ranked differently. Thus, it is clear that the suggested approach takes into account the contextual element and produces results by adjusting the evaluation using data that is concurrently collected by both C-PiFS and Crq-ROFS

3. Compared with the current approaches in the Complex Picture fuzzy environment, and Circular rung orthopair fuzzy fuzzy environment, the presented method offers significantly more information for handling data ambiguities. It provides a more accurate and precise statement of the object-related knowledge. As such, it is an invaluable mechanism for handling vague and perplexing information when making decisions.

4. We have thoroughly compared the performance of these operators using a variety of metrics, including accuracy, precision, and recall, in order to show the advantages of our suggested mean operators over current methods like Akram *et al.* [29], Qu *et al.* [30], Khan *et al.* [31], Liu *et al.* [32], and Özer [33]. Our evaluation's findings demonstrate that our suggested mean operator continually performs better than the other strategies taken into account across all the measures examined, offering compelling evidence for its application in real-world decision-making situations.

An overview of the ideal score values and the order in which the alternatives are ranked is provided in Tables 15, 16 and 17. Looking at these tables, we can see that the optimal choice agrees with the outcomes of our suggested strategy. This demonstrates the consistency and efficacy of our suggested strategy when compared to the marketplace's most advanced techniques. It's also critical to remember that our suggested approach's computing process is different from previous approaches used in different contexts. But when it comes to making decisions, the outcomes of our suggested strategy are more in line with actuality. This is essentially one of the key factors in making decisions that our method considers is the consistent degree of prioritization throughout the points of view in pairings. Our suggested method is more sensible and useful in real-world situations because of this aspect.

5. Conclusively, the operators that have been suggested consider the decision maker's parameters, λ and β, thus providing them with an increased range of choices. The reason for this is that the decision-maker can choose the option that most effectively fits their requirements and preferences due to the alternative's ratings varying depending on the parameterization values of λ, and β. With the help of this characteristic of the suggested operators, the decision-maker has greater control and autonomy over the entire procedure.Tables 15, 16 and 17 show that solutions Akram *et al.* [29], Qu *et al.* [30], Khan *et al.* [31], Liu *et al.* [32], and Özer [33] are insufficient to solve the problem since they don't meet the standards of the CrC-PiFS.

6. Tables 16 and 17 make it clear that different alternatives receive different scores depending on the results of characteristics λ and β. The alternatives are still ranked in the same order. The ranking results show that $C_4 \succeq C_2 \succeq C_1 \succeq C_3$ for $\lambda = 1, 2, 3, 4, 5$. Variations in parameters do not significantly affect the ranking order of the alternatives, which is a strong indication of the robustness and stability of the suggested technique. This result reinforces the applicability of the suggested approach for real-world decision-making settings and boosts our confidence in its dependability.

Comparative analysis for CrC-PiF environment

In order to evaluate the effectiveness of the proposed aggregation operators, this subsection contrasts several existing aggregation operators. The existing operators used for comparison in this study include: Kadyan *et al.* [34], who applied picture fuzzy set theory to reliability and risk analysis of complex industrial systems; Mahmood *et al.* [35], who developed complex picture fuzzy soft power aggregation operators for multi-attribute decision making; Garg [36], who established picture fuzzy aggregation operators and their applications; Wei [37], who explored picture fuzzy aggregation operators and their applications; and Wei [38], who proposed picture fuzzy Hamacher aggregation operators. Then, using the data from Tables 18, 19, 20, 25, 26, 27 and 28, we perform a comparative analysis, which are detailed in Tables 26, 27 and 28. Our suggested techniques (CrC-$PiFWAM$ and CrC-$PiFWGM$) has change the no ranking as the operators defined by Qu *et al.* [30] and Ali *et al.* [20]. It is evident from Tables 26, 27 and 28 that the scores assigned to various alternatives vary based on the variables λ and β. For $\lambda = 1$, 2, 3, 4, 5 the alternatives remain ordered in the same sequence, $C_4 \succeq C_2 \succeq C_1 \succeq C_3$. The ranking order of the alternatives remains unaffected by changes in parameters, indicating the stability and robustness of the proposed method. This outcome increases our trust in the suggested approach's reliability and validates its application for real-world decision-making contexts. A more thorough look at the previous discussion makes it clear that the recommended operators include the current aggregation operators in different situations. This realization highlights the methodology's adaptability and versatility, as it can appropriately respond to various conditions and produce more sophisticated answers. The suggested technique offers a more comprehensive framework for handling decision-making situations that can satisfy a wider range of preferences and requirements by combining the current aggregation operators. Tables 26, 27 and 28 present a comparative analysis of the suggested methodology with other alternative approaches, highlighting their distinct features. This comparison helps to demonstrate how effective the suggested method is at managing a variety of decision-making scenarios.

Significance of the proposed model

1. CrC-PiF arithmetic and geometric mean aggregation operators offer a high degree of flexibility when aggregating confusing data. These operators have been effectively applied in domains such as expert systems, data fusion, pattern recognition, and decision-making. Their ability to handle complicated and unpredictable information makes them suitable for managing ambiguous and inaccurate real-world situations. Their unique approach to data aggregation yields findings that are more reliable and accurate, assisting decision-makers in a range of applications in making better decisions.

2. Stock Marketing risk assessment involves numerous variables and interdependent factors that are often difficult to clearly identify. By applying generalized fuzzy logic and mathematical modeling, these methods can capture the inherent uncertainty and imprecision in the decision-making process, leading to more reliable and accurate evaluations of Stock Marketing risks. This enhanced precision allows organizations to perform better comparisons and make more informed decisions when selecting the most effective Stock Marketing strategies and protective measures.

3. Tables 26, 27 and 28 make it clear that different alternatives receive different scores depending on the results of characteristics λ and β. The alternatives are still ranked in the same order $C_4 \succeq C_3 \succeq C_2 \succeq C_1$ for $\lambda = 1, 2, 3, 4, 5$. Variations in parameters do not significantly affect the ranking order of the alternatives, which is a strong indication of the robustness and stability of the suggested technique. This result reinforces the applicability of the suggested approach for real-world decision-making settings and boosts our confidence in its dependability.

4. Upon meticulous examination, it has been ascertained that the proposed operators consider the parameters of the DMs, λ and β. These parameters afford DMs a wide array of options from which to select, given the different scores assigned to each alternative according to various parametric values of λ and β. Consequently, the suggested operators give DMs the freedom to choose alternatives that correspond with their particular preferences, depending on how the alternatives are assessed using various values of λ and β.

Table 29 (a) Comparative study of proposed and existing methods

Methods	Whether deals with group decision making?	Methods for aggregation	Criteria weights calculation	Generality and flexibility of the aggregation operators
Akram *et al.* [29]	Yes	Nil	Direct	Nil
Qu *et al.* [30]	Yes	Nil	Direct	Low
Khan *et al.* [31]	Yes	Nil	Direct	Nil
Liu *et al.* [32]	Yes	Nil	Direct	Nil
Özer [33]	Yes	Nil	Direct	Nil
Kadyan *et al.* [34]	Yes	Nil	Direct	Low
Mahmood *et al.* [35]	Yes	Nil	Direct	Nil
Garg [36]	Yes	Nil	Direct	Nil
Wei [37]	Yes	Nil	Direct	Nil
Wei [38]	Yes	Nil	Direct	High
Ali *et al.* [20]	Yes	Nil	Direct	Low
Proposed Method	Yes	TOPSIS	CRITIC	High

Table 29 (b) Comparative study of proposed and existing methods

Methods	Whether captures Circular information?	Whether captures Four parameters information?	Ranking of alternatives
Akram *et al.* [29]	No	No	No ranking
Qu *et al.* [30]	No	No	No ranking
Khan *et al.* [31]	No	No	No ranking
Liu *et al.* [32]	No	No	No ranking
Özer [33]	No	No	No ranking
Kadyan *et al.* [34]	No	No	No ranking
Mahmood *et al.* [35]	No	No	No ranking
Garg [36]	No	No	No ranking
Wei [37]	No	No	No ranking
Wei [38]	No	No	No ranking
Ali *et al.* [20]	No	No	No ranking
Proposed Method	Yes	Yes	Yes

7. Conclusion

The purpose of this study, we discussed the notion of complex operational laws for CrC-PiFNs. Based on complex operational laws under CrC-PiFNs, we offered the idea of Circular Complex PiSF weighted arithmetic mean aggregation operator (CrC-PiFWAM), Circular Complex PiSF ordered weighted arithmetic mean aggregation operator (CrC-PiFOWAM), Circular Complex PiSF weighted geometric mean aggregation operator (CrC-PiFWGM) and Circular Complex PiSF ordered weighted geometric mean aggregation operator (CrC-PiFOWGM). Additionally, we go into extensive detail on the newly suggested multi-attribute group decision process that is based on the complex picture fuzzy preference environment. The paper also looks into and analyses the different advantageous aspects of these operators. Lastly, we offer a framework that takes into account various permutations of the parameters λ and β in order to tackle decision-making problems. We used the artificial intelligence tools in education system problem to further verified the feasibility of the approach. The decision-maker has a variety of opportunities for decision evaluation using this technique, which also increases the adaptability of the suggested operators. while comparing the suggested operators and their related methodologies with various available operators, it becomes clear that the former provides the decision-maker with a more favorable, genuine, and consistent strategy while gathering data to make choices. Considering five important factors, the current investigation integrated feedback from specialists in five distinct choices. Future studies could broaden the scope by include more choices and qualities in order to confirm the findings of this one. According to the study, the options' ranking order should stay the same for different values of λ, and β. Furthermore, this technique is widely applicable and can be used to tackle a range of decision-making difficulties such as medical assessment, domestic aviation assessments, handling medical waste, and evaluation of medical waste treatment methods.

Furthermore, we highlight some meaningful future study options below:

(i) We will introduce a novel decision-making method that combines the WASPAS method with our proposed models. We will applying this method to a real multiple conditional attributes in addition to multiple decision attributes problems of selecting the best building shape, risk assessment, dynamic decision making, business and marketing management, human resource management. Through comparative analysis, parameter analysis, and experimental analysis, we will demonstrating the effectiveness and feasibility of our proposed methods.

(ii) We will establish a three way decision model according to our novel approach of CrC-PiFWAM and CrC-PiFWGM from multisource fuzzy information and their applications to conflict analysis problems.

(iii) We will apply the suggested methodology to large-scale group decision-making and base the attribute weights on user feedback gathered through social media.

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Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, *8*(3), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X
[2] Atanassov, K. T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, *20*(1), 87–96. https://doi.org/10.1016/S0165-0114(86)80034-3
[3] Yager, R. R. (2013). Pythagorean fuzzy subsets. *IEEE Transactions on Fuzzy Systems*, *21*(1), 57–61. https://doi.org/10.1109/IFSA-NAFIPS.2013.6608375
[4] Yager, R. R. (2017). Generalized orthopair fuzzy sets. *IEEE Transactions on Fuzzy Systems*, *25*(5), 1222–1230. https://doi.org/10.1109/TFUZZ.2016.2604005
[5] Cuong, B. C. (2014). Picture fuzzy sets. *Computer Science and Cybernetics*, *30*, 409–420. https://doi.org/10.15625/1813-9663/30/4/5032
[6] Mahmood, T., Ullah, K., Khan, Q., & Jan, N. (2019). An approach towards decision making and medical diagnosis problems using the concept of spherical fuzzy sets. *Neural Computing and Applications*, *31*, 7041–7053. https://doi.org/10.1007/s00521-018-3521-2
[7] Ullah, K., Mahmood, T., & Jan, N. (2018). Similarity measures for t-spherical fuzzy sets with applications in pattern recognition. *Symmetry*, *10*(6), 193. https://doi.org/10.3390/sym10060193
[8] Ramot, D., Milo, R., Friedman, M., & Kandel, A. (2002). Complex fuzzy sets. *IEEE Transactions on Fuzzy Systems*, *10*(2), 171–186. https://doi.org/10.1109/91.995119
[9] Alkouri, A., & Salleh, A. R. (2012). Complex intuitionistic fuzzy sets. *AIP Conference Proceedings*, *1482*, 464–470. https://doi.org/10.1063/1.4757515
[10] Ullah, K., Mahmood, T., Ali, Z., & Jan, N. (2020). On some distance measures of complex pythagorean fuzzy sets and their applications in pattern recognition. *Complex Intelligent Systems*, *6*, 15–27. https://doi.org/10.1007/s40747-019-0103-6
[11] Liu, P., Mahmood, T., & Ali, Z. (2020). Complex q-rung orthopair fuzzy aggregation operators and their applications in multi-attribute group decision making. *Information*, *11*(1), 5. https://doi.org/10.3390/info11010005
[12] Akram, M., Bashir, A., & Garg, H. (2020a). Decision-making model under complex picture fuzzy hamacher aggregation operators. *Computational and Applied Mathematics*, *39*(3), 226. https://doi.org/10.1007/s40314-020-01251-2
[13] Akram, M., Kahraman, C., & Zahid, K. (2021). Group decision-making based on complex spherical fuzzy vikor approach. *Knowledge-Based Systems*, *216*, 106793. https://doi.org/10.1016/j.knosys.2021.106793
[14] Ali, Z., Mahmood, T., & Yang, M. S. (2020). Complex t-spherical fuzzy aggregation operators with application to multi-attribute decision making. *Symmetry*, *12*(8), 1311. https://doi.org/10.3390/sym12081311
[15] Atanassov, K. T. (2020). Circular intuitionistic fuzzy sets. *Journal of Intelligent Fuzzy Systems*, *39*, 5981–5986. https://doi.org/10.1109/ACCESS.2024.3472215
[16] Bozyigit, M. C., Olgun, M., & Enver, M. (2023). Circular pythagorean fuzzy sets and applications to multi-criteria decision making. *Informatica*, *34*, 713–742. https://doi.org/10.15388/23-INFOR529

[17] Yusoff, B., Kilicman, A., Pratama, D., & Hasni, R. (2023). Circular q-rung orthopair fuzzy set and its algebraic properties. *Malaysian Journal of Mathematical Sciences*, *17*, 363–378. https://doi.org/10.47836/mjms.17.3.08
[18] Ali, Z., & Yang, M. S. (2024). On circular q-rung orthopair fuzzy sets with dombi aggregation operators. *Symmetry*, *16*, 24. https://doi.org/10.3390/sym16030260
[19] Kahraman, C., & Otay, I. (2021). Extension of vikor method using circular intuitionistic fuzzy sets. *INFUS*, *2*, 24–26. https://doi.org/10.1007/978-3-030-85577-2-6
[20] Hussain, A., Ullah, K., Tehreem, T., Senapati, T., & Moslem, S. (2023). A robust decision-making approach for supplier selection using complex picture fuzzy information. *IEEE Access*, *11*, 91807–91830. https://doi.org/10.1109/ACCESS.2023.3308030
[21] Shoaib, M., Mahmood, W., Xin, Q., Techier, F., & Tawfiq, F. M. O. (2022). Certain operations on complex picture fuzzy graphs. *IEEE Access*, *10*, 114284–114296. https://doi.org/10.1109/ACCESS.2022.3216615
[22] Liu, P., Akram, M., & Bashir, A. (2021a). Extensions of power aggregation operators for decision making based on complex picture fuzzy knowledge. *Journal of Intelligent Fuzzy Systems*, *40*, 10461–10476. https://doi.org/10.3233/JIFS-201385
[23] Dhumras, H., Shukla, V., Bajaj, R. K., Driss, M., & Boulila, W. (2024). On similarity measures of complex picture fuzzy pattern recognition. *IEEE Access*, *12*, 83104–83117. https://doi.org/10.1109/ACCESS.2024.3412755
[24] Khan, Z., Hussain, F., Rahim, T., Jan, R., & Boulaaras, S. (2024a). Distance measures and applications to decision making, medical diagnosis, and pattern recognition under complex picture fuzzy sets. *European Physical Journal Plus*, *139*, 243. https://doi.org/10.1140/epjp/s13360-024-04996-5
[25] Khan, S. U., Al-Sabri, E. H. A., Ismail, R., Mohammed, M., Mohammed, S., Hussain, S., & Mehmood, A. (2023). Prediction model of a generative adversarial network using the concept of complex picture fuzzy soft information. *Symmetry*, *15*, 577. https://doi.org/10.3390/sym15030577
[26] Tanoli, M. N. K., Gulistan, M., Amin, F., & Al-Shamiri, M. M. (2024). Decision-making model based on complex cubic picture fuzzy information and geometric aggregation operators with applications. *Complex Intelligent Systems*, *10*, 1801–1843. https://doi.org/10.1007/s40747-023-01217-x
[27] Ali, Z., Mahmood, T., & Yang, M.-S. (2023). Aczél–alsina power aggregation operators for complex picture fuzzy sets with application in multi-attribute decision making. *Symmetry*, *15*, 651. https://doi.org/10.3390/sym15030651
[28] Zavadskas, E. K., Turskis, Z., Antucheviciene, J., & Zakarevicius, A. (2012). Optimization of weighted aggregated sum product assessment. *Elektronika ir Elektrotechnika*, *122*(6), 3–6. https://doi.org/10.5755/j01.eee.122.6.1810
[29] Akram, M., Bashir, A., & Garg, H. (2020b). Decision-making model under complex picture fuzzy hamacher aggregation operators. *Computational and Applied Mathematics*, *39*, 226. https://doi.org/10.1007/s40314-020-01251-2
[30] Qu, J., Nasir, A., Khan, S. U., Nonlaopon, K., & Rahman, G. (2022). An innovative decision-making approach based on correlation coefficients of complex picture fuzzy sets and their applications in cluster analysis. *Computational Intelligence and Neuroscience*. https://doi.org/10.1155/2022/7389882
[31] Khan, Z., Hussain, F., Rahim, T., Jan, R., & Boulaaras, S. (2024b). Distance measures and their applications to decision making, medical diagnosis, and pattern recognition under complex picture fuzzy sets. *European Physical Journal Plus*, *139*, 243. https://doi.org/10.1140/epjp/s13360-024-04996-5

[32] Liu, P., Akram, M., & Bashir, A. (2021b). Extensions of power aggregation operators for decision making based on complex picture fuzzy knowledge. *Journal of Intelligent Fuzzy Systems*, *40*(5), 1–22. https://doi.org/10.3233/JIFS-202889

[33] Özer, Ö. (2022). Hamacher prioritized aggregation operators based on complex picture fuzzy sets and their applications in decision-making problems. *Journal of Innovative Research in Mathematical and Computational Sciences*, *1*(1), 33–54.

[34] Kadyan, M. S., Bura, J., & Kumar, J. (2024). Application of picture fuzzy set theory to reliability and risk analysis of complex industrial systems. *Life Cycle Reliability and Safety Engineering*, *13*, 293–307. https://doi.org/10.1007/s41872-024-00262-w

[35] Mahmood, T., Ali, Z., & Aslam, M. (2022). Applications of complex picture fuzzy soft power aggregation operators in multi-attribute decision making. *Scientific Reports*, *12*, 12415. https://doi.org/10.1038/s41598-022-20239-y

[36] Garg, H. (2017). Some picture fuzzy aggregation operators and their applications to multi-attribute decision-making. *Arab Journal of Science and Engineering*, *42*, 5275–5290. https://doi.org/10.1007/s13369-017-2625-9

[37] Wei, G. (2017). Picture fuzzy aggregation operators and their application to multiple attribute decision making. *Journal of Intelligent Fuzzy Systems*, *33*, 713–724. https://doi.org/10.3233/JIFS-161798

[38] Wei, G. (2018). Picture fuzzy hamacher aggregation operators and their application to multiple attribute decision making. *Fundamenta Informaticae*, *157*, 271–320. https://doi.org/10.3233/FI-2018-1628